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# THE FINITE GROUP ACTION AND THE EQUIVARIANT DETERMINANT OF ELLIPTIC OPERATORS III

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## Abstract

Let  $M$  be an even-dimensional closed oriented manifold and  $g$  a periodic automorphism of  $M$  of order  $p$ . In this paper, under the assumption that the fixed points of  $g^k$  ( $1 \leq k \leq p-1$ ) are isolated, a calculation formula is provided for the homomorphism  $I_D : \mathbb{Z}_p \rightarrow \mathbb{R}/\mathbb{Z}$  defined in [6] for equivariant twisted signature operators  $D$  over  $M$ . The formula gives a new method to study the periodic automorphisms of oriented manifolds. As examples of the application of the formula, results about the existence of the cyclic group action for 2,4,6-dimensional closed oriented manifolds are obtained.

## 1. Introduction

Let  $M$  be a  $2m$ -dimensional closed oriented manifold and  $G$  a finite group acting on  $M$ . We assume that the action of  $G$  is effective and orientation-preserving. Let  $g$  be an element of  $G$  of order  $p \geq 2$  and  $\mathbb{Z}_p = \langle g \rangle$  the cyclic group generated by  $g$ . In this paper, we set the following assumption.

**Assumption 1.1.** *Some  $g^k$  ( $1 \leq k \leq p-1$ ) has a fixed point, and any fixed point of  $g^k$  is isolated for  $1 \leq k \leq p-1$  if  $g^k$  has a fixed point.*

In [6] we introduce a group homomorphism  $I_D$  by using an elliptic operator  $D$  adapted to a geometric structure of a manifold, and in [7] we apply this homomorphism to the existence problem of finite group actions on almost complex manifolds under the assumption above.

Let  $D$  be a  $G$ -equivariant elliptic operator. Then a homomorphism  $I_D$  from  $G$  to  $\mathbb{R}/\mathbb{Z}$  is defined by

$$I_D(g) = \frac{1}{2\pi\sqrt{-1}} \log \det(D, g) \in \mathbb{R}/\mathbb{Z}$$

for  $g \in G$ , where  $\det(D, g)$  is defined by

$$\det(D, g) = \det(g| \ker D) / \det(g| \operatorname{coker} D) \in S^1 \subset \mathbb{C}^*$$

(see [6] Definition 2.1). Then as we see in [6] (3) the next equality holds

$$(1) \quad I_D(g) \equiv \frac{p-1}{2p} \operatorname{Ind}(D) - \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - \xi_p^{-k}} \operatorname{Ind}(D, g^k) \pmod{\mathbb{Z}}$$

where  $\text{Ind}$  is the Atiyah-Singer index (see [3]) and  $\xi_p$  is the primitive  $p$ -th root of unity defined by  $\xi_p = e^{2\pi\sqrt{-1}/p}$ .

We can express the value  $I_D(g)$  by the fixed point data of the  $g^k$ -action ( $1 \leq k \leq p-1$ ) by using the equality (1) and the fixed point formula of Atiyah-Segal-Singer [2], [3]. Since  $I_D$  is a homomorphism, the equalities  $I_D(g^z) = zI_D(g)$ ,  $I_D(gh) = I_D(g) + I_D(h)$  hold for any  $g, h \in G$  and any integer  $z$  because  $\mathbb{R}/\mathbb{Z}$  is an abelian group. These properties of  $I_D$  impose conditions on the fixed point data and a calculation formula to obtain the precise value of  $I_D$  can be used to examine the existence of a finite group action on  $M$ .

When  $M$  is a  $2m$ -dimensional closed almost complex manifold and the  $g$ -action preserves the almost complex structure of  $M$ , we give a formula to calculate  $I_D(g)$  precisely for the  $\otimes_{j=1}^m (\otimes_{\ell_j}^j (\wedge_{\mathbb{C}}^j TM))$ -valued Dolbeault operator  $D$  over  $M$  in [7] Theorem 2.2. Though the formula is useful for closed almost complex manifolds, we need a formula to calculate the precise value of  $I_D(g)$  for equivariant elliptic operators  $D$  which is adapted for closed oriented manifolds, which do not necessarily have almost complex structure.

In [9] Zagier gives a formula which enables us to calculate the precise value of  $I_D(g)$  for the equivariant non-twisted signature operator  $D$  under Assumption 1.1.

**DEFINITION 1.2.** For a real number  $x$ , a real number  $((x))$  is defined by

$$((x)) := \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer} \\ 0 & \text{if } x \text{ is an integer} \end{cases}$$

where  $[x]$  is the Gauss' symbol. Note that  $((x+a)) = ((x))$  for an integer  $a$  and that  $((-x)) = -((x))$  because  $[-x] = -[x] - 1$  if  $x$  is not an integer.

**Theorem 1.3** ([9] p.103). *Let  $p$  be a natural number which is greater than 1 and  $a_1, \dots, a_n$  natural numbers which are prime to  $p$ . Then the following equality holds:*

$$(-\sqrt{-1})^n \sum_{k=1}^{p-1} \cot \frac{\pi a_1 k}{p} \cdots \cot \frac{\pi a_n k}{p} = 2^n p \sum_{\substack{1 \leq k_1, \dots, k_n < p \\ p | (a_1 k_1 + \dots + a_n k_n)}} \left( \left( \frac{k_1}{p} \right) \right) \cdots \left( \left( \frac{k_n}{p} \right) \right).$$

As we see later (see (4)) the left hand side of the equality above is equal to the sum of indices for the equivariant non-twisted signature operator, and the equality implies that the sum is a rational number.

In this paper, generalizing the Zagier's formula, we give a formula to calculate the precise value of  $I_D(g)$  for equivariant twisted signature operators  $D$  over  $2m$ -dimensional closed oriented manifolds to examine the existence of a finite group action on the manifolds.

## 2. Rotation angles of periodic automorphisms

Let  $M$  be a  $2m$ -dimensional closed oriented manifold and  $p$  a natural number which is greater than 1. Assume that the cyclic group  $\mathbb{Z}_p = \langle g \rangle$  acts on  $M$ . Under Assumption 1.1, let  $\Omega$  be the union of the fixed points of  $g^k$  for  $1 \leq k \leq p-1$  and suppose that the image  $\pi(\Omega)$  consists of  $b$  points  $y_1, \dots, y_b \in M/\mathbb{Z}_p$  where  $\pi : M \rightarrow M/\mathbb{Z}_p$  is the projection. In this paper, the  $\mathbb{Z}_p$ -action is called the  $\mathbb{Z}_p$ -action of isotropy orders  $(p_1, \dots, p_b)$  if the isotropy group at a point  $q_i \in \pi^{-1}(y_i)$  ( $1 \leq i \leq b$ ) is the cyclic group of order  $p_i$ . Then for  $1 \leq i \leq b$  the isotropy group at any points in  $\pi^{-1}(y_i)$  is the cyclic group of order  $p_i$  generated by  $g^{r_i}$

where  $r_i = p/p_i$  and  $\pi^{-1}(y_i)$  consists of  $r_i$  points  $q_i, g \cdot q_i, \dots, g^{r_i-1} \cdot q_i$ .

REMARK 2.1. Let  $e(M)$  be the Euler number of  $M$ . Then since  $\mathbb{Z}_p$  acts freely on the punctured manifold  $M_0 = M \setminus \{\cup_{i=1}^b \pi^{-1}(y_i)\}$ , the next equality holds:

$$(2) \quad e(M) \equiv \sum_{i=1}^b r_i \pmod{p}.$$

Let  $\xi_{p_i}^{\tau_{ij}}, \xi_{p_i}^{-\tau_{ij}}$  ( $1 \leq j \leq m$ ) be the eigenvalues of  $g^{r_i}|_{T_{q_i}M}$ , and

$$\vec{u}_{ij} - \sqrt{-1}\vec{v}_{ij}, \vec{u}_{ij} + \sqrt{-1}\vec{v}_{ij} \quad (\vec{u}_{ij}, \vec{v}_{ij} \in T_{q_i}M \simeq \mathbb{R}^{2m})$$

the corresponding eigenvectors such that

$$\omega_{q_i}(\vec{u}_{i1} \wedge \vec{v}_{i1} \wedge \dots \wedge \vec{u}_{im} \wedge \vec{v}_{im})$$

is positive with respect to a positive  $2m$ -form  $\omega$  where  $\tau_{ij}$  ( $1 \leq j \leq m$ ) are natural numbers such that  $1 \leq \tau_{ij} \leq p_i - 1$  (see ([1] p.473). Since the eigenvalues of  $g^{r_i\kappa}|_{T_{q_i}M}$  for  $1 \leq \kappa \leq p_i - 1$  are  $\xi_{p_i}^{\kappa\tau_{ij}}, \xi_{p_i}^{-\kappa\tau_{ij}}$  ( $1 \leq j \leq m$ ), it follows from Assumption 1.1 that  $\tau_{ij}$  is prime to  $p_i$  for  $1 \leq i \leq b, 1 \leq j \leq m$ . In this paper the set  $\{\tau_{ij}\}$  ( $1 \leq j \leq m$ ) is called the rotation angle of  $g^{r_i}$  around the points in  $\pi^{-1}(y_i)$  and the set  $\{\tau_{ij}\}$  ( $1 \leq i \leq b, 1 \leq j \leq m$ ) is called the rotation angle of  $g$ . The rotation angle of  $g$  is also expressed as

$$((\tau_{11}, \dots, \tau_{1m}), \dots, (\tau_{b1}, \dots, \tau_{bm})) .$$

Note that the equality

$$g^{r_i} \cdot (\vec{u}_{ij} - \sqrt{-1}\vec{v}_{ij}) = \xi_{p_i}^{\tau_{ij}} (\vec{u}_{ij} - \sqrt{-1}\vec{v}_{ij})$$

implies the equality

$$(g^{r_i} \cdot \vec{u}_{ij}, g^{r_i} \cdot \vec{v}_{ij}) = (\vec{u}_{ij}, \vec{v}_{ij}) \begin{pmatrix} \cos \frac{2\pi\tau_{ij}}{p_i} & -\sin \frac{2\pi\tau_{ij}}{p_i} \\ \sin \frac{2\pi\tau_{ij}}{p_i} & \cos \frac{2\pi\tau_{ij}}{p_i} \end{pmatrix}$$

and that  $\{\tau_{i1}, \dots, p_i - \tau_{ij}, \dots, p_i - \tau_{ik}, \dots, \tau_{im}\}$  ( $j \neq k$ ) is also the rotation angle of  $g^{r_i}$  when  $m \geq 2$  because

$$\omega_{q_i}(\vec{u}_{i1} \wedge \vec{v}_{i1} \wedge \dots \wedge \vec{v}_{ij} \wedge \vec{u}_{ij} \wedge \dots \wedge \vec{v}_{ik} \wedge \vec{u}_{ik} \wedge \dots \wedge \vec{u}_{im} \wedge \vec{v}_{im})$$

is also positive. Hence if  $p_i$  is even and some  $\tau_{ij}$  equals  $p_i/2$ ,  $\{\tau_{i1}, \dots, p_i - \tau_{ik}, \dots, \tau_{im}\}$  is also the rotation angle of  $g^{r_i}$  for  $k \neq j$  and therefore we can assume that  $1 \leq \tau_{ij} \leq p_i/2$ .

Corresponding to the irreducible representations of the rotation group, complex  $\mathbb{Z}_p$ -vector bundles  $E^{\ell_1, \dots, \ell_m}$  are defined for non-negative integers  $\ell_1, \dots, \ell_m$  as follows:

$$E^{\ell_1, \dots, \ell_m} = \otimes_{j=1}^m \left( \otimes_{\mathbb{C}}^{\ell_j} \left( \wedge_{\mathbb{C}}^j (TM \otimes \mathbb{C}) \right) \right),$$

and the  $E^{\ell_1, \dots, \ell_m}$ -valued signature operator  $D_{\ell_1, \dots, \ell_m}$  is defined. Note that  $E^{0, \dots, 0}$  is a trivial line bundle and  $D_{0, \dots, 0}$  is the equivariant non-twisted signature operator. Suppose that the rotation angle of  $g^{r_i}$  around the points  $q_i$  in  $\pi^{-1}(y_i)$  ( $1 \leq i \leq b$ ) is  $\{\tau_{ij}\}$ . Then the fixed point set of  $g^k$  ( $1 \leq k \leq p-1$ ) exists if and only if  $k$  equals  $r_i\kappa$  for  $1 \leq i \leq b, 1 \leq \kappa \leq p_i - 1$  and it follows from the signature theorem and the  $G$ -signature theorem that

$$(3) \quad \text{Ind}(D_{\ell_1, \dots, \ell_m}) = 2^m \text{Ch}(E^{\ell_1, \dots, \ell_m}) \widehat{\mathbf{L}}(M)[M] \quad \left( \widehat{\mathbf{L}}(M) = \prod_{j=1}^m \frac{x_j/2}{\tanh(x_j/2)} \right),$$

$$(4) \quad \text{Ind}(D_{\ell_1, \dots, \ell_m}, g^{r_i \kappa}) = \sum_{i=1}^b r_i \text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, g^{r_i \kappa}) \prod_{j=1}^m \left( -\sqrt{-1} \cot \frac{\pi \kappa \tau_{ij}}{p_i} \right)$$

where  $\sigma_j(x_1^2, \dots, x_m^2)$  for the  $j$ -th elementary symmetric function  $\sigma_j$  is the  $j$ -th Pontrjagin class of  $M$  (see [3] (6.5), (6.6), [5] Theorem 13.9),  $E_{q_i}^{\ell_1, \dots, \ell_m}$  is the fiber of  $E^{\ell_1, \dots, \ell_m}$  at  $q_i$  and  $\text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, g^{r_i \kappa})$  is the trace of the linear transformation  $g^{r_i \kappa}$  of the fiber  $E_{q_i}^{\ell_1, \dots, \ell_m}$  (see [3] (6.12), (6.19), [4] Theorem 2.5), and hence we have

$$(5) \quad \begin{aligned} & I_{D_{\ell_1, \dots, \ell_m}}(g) \\ & \equiv \frac{p-1}{p} 2^{m-1} \text{Ch}(E^{\ell_1, \dots, \ell_m}) \widehat{\mathbf{L}}(M)[M] \\ & \quad - \frac{1}{p} \sum_{i=1}^b r_i \sum_{\kappa=1}^{p_i-1} \frac{1}{1 - \xi_{p_i}^{-\kappa}} \text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, g^{r_i \kappa}) \prod_{j=1}^m \left( -\sqrt{-1} \cot \frac{\pi \kappa \tau_{ij}}{p_i} \right) \pmod{\mathbb{Z}}. \end{aligned}$$

REMARK 2.2. Since the Chern classes of  $TM \otimes \mathbb{C}$  is expressed by the Pontrjagin classes of  $TM$ , the Chern character  $\text{Ch}(E^{\ell_1, \dots, \ell_m})$  is expressed by the Pontrjagin classes of  $TM$  (see Proposition 2.6). Hence the equality (3) above implies that  $\text{Ind}(D_{\ell_1, \dots, \ell_m}) = 0$  if  $m$  is odd.

REMARK 2.3. Assume that  $p_i = 2$  for some  $i$ . Then since  $\kappa = \tau_{ij} = 1$ , we have

$$\prod_{j=1}^m \left( -\sqrt{-1} \cot \frac{\pi \kappa \tau_{ij}}{p_i} \right) = \prod_{j=1}^m \left( -\sqrt{-1} \cot \frac{\pi}{2} \right) = 0$$

in the equalities (4), (5).

DEFINITION 2.4. Let  $p$  be a natural number which is greater than 1 and  $a, b$  integers where  $b$  is prime to  $p$ . Then using the Euler's function  $\varphi$ , we define integers  $\langle a \rangle, \bar{b}$  as follows:

$$\begin{aligned} \langle a \rangle &= a - p \left\lfloor \frac{a}{p} \right\rfloor \iff 0 \leq \langle a \rangle \leq p-1, \langle a \rangle \equiv a \pmod{p}, \\ \bar{b} &= \langle b^{\varphi(p)-1} \rangle \iff b\bar{b} \equiv 1 \pmod{p}, \quad 1 \leq \bar{b} \leq p-1. \end{aligned}$$

In this paper, we call  $\bar{b}$  the mod.  $p$ -inverse of  $b$ . Note that  $\langle a+p \rangle = \langle a \rangle$ ,  $\overline{b+p} = \bar{b}$ ,  $\bar{\bar{b}} \equiv b \pmod{p}$  because  $(\bar{b}-b)\bar{b} \equiv 0 \pmod{p}$  and that  $\langle a_1 \rangle \cdot \langle a_2 \rangle \equiv \langle a_1 a_2 \rangle$ ,  $\bar{b}_1 \cdot \bar{b}_2 \equiv \overline{b_1 b_2} \pmod{p}$  because  $(\bar{b}_1 \cdot \bar{b}_2 - \overline{b_1 b_2}) b_1 b_2 \equiv 0 \pmod{p}$  where  $b_1, b_2$  are prime to  $p$ . Note moreover that the equality  $\langle a_1 \rangle + \langle a_2 \rangle \equiv \langle a_1 + a_2 \rangle \pmod{p}$  holds but the equality  $\bar{b}_1 + \bar{b}_2 \equiv \overline{b_1 + b_2} \pmod{p}$  does not hold.

REMARK 2.5. Assume that a natural number  $z$  is prime to  $p$ . Then we have  $(g^{kz})^{\bar{z}} = g^k$  for  $1 \leq k \leq p-1$ , and hence the fixed point set of  $g^{kz}$  coincides with that of  $g^k$  and the rotation angle of  $g^{r_i z}$  is equivalent to  $\{z\tau_{ij}\} \pmod{p_i}$  if the rotation angle of  $g^{r_i}$  is  $\{\tau_{ij}\}$ .

**Proposition 2.6.** Suppose that the total Chern class of  $TM \otimes \mathbb{C}$  equals  $\prod_{j=1}^m (1 - x_j^2)$  and hence the  $j$ -th Pontrjagin class  $p_j(TM)$  of  $M$  equals  $\sigma_j(x_1^2, \dots, x_m^2)$  where  $\sigma_j$  is the  $j$ -th

elementary symmetric function (see [5] p.228). Then we have

$$(6) \quad \text{Ch}(E^{\ell_1, \dots, \ell_m}) = \prod_{j=1}^m \sigma_j(e^{x_1}, e^{-x_1}, \dots, e^{x_m}, e^{-x_m})^{\ell_j}.$$

Proof. Since the total Chern class of  $TM \otimes \mathbb{C}$  equals  $\prod_{j=1}^m (1 + x_j)(1 - x_j)$ , we have

$$\text{Ch}(\wedge_{\mathbb{C}}^j(TM \otimes \mathbb{C})) = \sigma_j(e^{x_1}, e^{-x_1}, \dots, e^{x_m}, e^{-x_m}).$$

Hence it follows that

$$\text{Ch}(E^{\ell_1, \dots, \ell_m}) = \prod_{j=1}^m \sigma_j(e^{x_1}, e^{-x_1}, \dots, e^{x_m}, e^{-x_m})^{\ell_j}. \quad \square$$

EXAMPLE 2.7. Assume that  $m = 2$  and let  $p_1$  be the first Pontrjagin number of  $M$ . Then we have

$$2^2 \widehat{\mathbf{L}}(M) = 2^2 \prod_{j=1}^2 \frac{x_j/2}{\tanh(x_j/2)} = 4 + \frac{1}{3} p_1(TM),$$

$$\text{Ch}(E^j) = \sigma_j(e^{x_1}, e^{-x_1}, e^{x_2}, e^{-x_2}) = \begin{cases} 4 + p_1(TM) & (j = 1, 3), \\ 6 + 2p_1(TM) & (j = 2) \\ 1 & (j = 0, 4) \end{cases},$$

where  $E^j = \wedge_{\mathbb{C}}^j(TM \otimes \mathbb{C})$ , and hence it follows from Proposition 2.6 that

$$\begin{aligned} \text{Ch}(\otimes^{\ell_j} E^j) &= \left( \text{Ch}(E^j) \right)^{\ell_j} = \begin{cases} 4^{\ell_j} + \ell_j 4^{\ell_j-1} p_1(TM) & (j = 1, 3), \\ 6^{\ell_j} + 2\ell_j 6^{\ell_j-1} p_1(TM) & (j = 2) \\ 1 & (j = 0, 4) \end{cases} \\ \implies \text{Ch}(E^{\ell_1, \ell_2}) &= \left( \text{Ch}(E^1) \right)^{\ell_1} \left( \text{Ch}(E^2) \right)^{\ell_2} \\ &= 4^{\ell_1} 6^{\ell_2} + 2 \cdot 4^{\ell_1-1} 6^{\ell_2-1} (3\ell_1 + 4\ell_2) p_1(M) \\ \implies \text{Ind} D_{\ell_1, \ell_2} &= 2^2 \text{Ch}(E^{\ell_1, \ell_2}) \widehat{\mathbf{L}}(M)[M] \\ &= 2 \cdot 4^{\ell_1} 6^{\ell_2-1} (3\ell_1 + 4\ell_2 + 1) p_1 = 2^{2\ell_1 + \ell_2} 3^{\ell_2-1} (3\ell_1 + 4\ell_2 + 1) p_1. \end{aligned}$$

DEFINITION 2.8. Let  $p$  be a natural number which is greater than 1. Then for any integer  $\lambda$ , a real number  $\rho_p(\lambda)$  is defined by

$$\rho_p(\lambda) = \xi_p^\lambda + \xi_p^{-\lambda}.$$

Clearly  $\rho_p$  has the following properties:

$$\rho_p(-\lambda) = \rho_p(\lambda), \quad \rho_p(0) = 2, \quad \rho_p(\lambda) \rho_p(\mu) = \rho_p(\lambda + \mu) + \rho_p(\lambda - \mu),$$

which implies that

$$(7) \quad \rho_p(\lambda_1) \rho_p(\lambda_2) \cdots \rho_p(\lambda_j) = \rho_p(\lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_j)$$

for integers  $\mu, \lambda_1, \lambda_2, \dots, \lambda_j$  where

$$\rho_p(\lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_j) = \sum_{\varepsilon_i = \pm 1 (2 \leq i \leq j)} \rho_p(\lambda_1 + \varepsilon_2 \lambda_2 + \cdots + \varepsilon_j \lambda_j).$$

We set that the binomial coefficient  $\binom{0}{0}$  and  $0^0$  are equal to 1 hereafter.

**Proposition 2.9.** For  $1 \leq i \leq b$ ,  $1 \leq j \leq m$ ,  $1 \leq k \leq p_i - 1$  set

$$C_{ij}^k = \sum_{r=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{m-j+2r}{r} \sigma_{j-2r}(\rho_{p_i}(k\tau_{i1}), \dots, \rho_{p_i}(k\tau_{im})) + \varepsilon_j \binom{m}{\frac{j}{2}}$$

where  $\sigma_{j-2s}$  is the  $j-2s$ -th elementary symmetric function and

$$\varepsilon_j = \frac{1 + (-1)^j}{2} = \begin{cases} 1 & (j : \text{even}) \\ 0 & (j : \text{odd}) \end{cases}.$$

Then we have

$$\text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, g^{r_i k}) = \prod_{j=1}^m (C_{ij}^k)^{\ell_j}.$$

**Proof.** Since the eigenvalues of  $g^{r_i k} | (T_{q_i} M \otimes \mathbb{C})$  are  $\lambda_1, \dots, \lambda_{2m}$  where  $\lambda_{2j-1} = \xi_{p_i}^{k\tau_{ij}}$ ,  $\lambda_{2j} = \xi_{p_i}^{-k\tau_{ij}}$  ( $1 \leq j \leq m$ ), we have

$$\begin{aligned} \text{Ch}(\wedge_{\mathbb{C}}^j (T_{q_i} M \otimes \mathbb{C}), g^{r_i k}) &= \text{Tr} \left( g^{r_i k} \Big| \wedge_{\mathbb{C}}^j (T_{q_i} M \otimes \mathbb{C}) \right) \\ &= \sum_{1 \leq i_1 < \dots < i_j \leq 2m} \lambda_{i_1} \cdots \lambda_{i_j} = \sigma_j(\xi_{p_i}^{k\tau_{i1}}, \xi_{p_i}^{-k\tau_{i1}}, \dots, \xi_{p_i}^{k\tau_{im}}, \xi_{p_i}^{-k\tau_{im}}) \\ &= \frac{1}{j!} \lim_{t \rightarrow 0} \left( \frac{d}{dt} \right)^j (1 + t \xi_{p_i}^{k\tau_{i1}})(1 + t \xi_{p_i}^{-k\tau_{i1}}) \cdots (1 + t \xi_{p_i}^{k\tau_{im}})(1 + t \xi_{p_i}^{-k\tau_{im}}) \\ &= \frac{1}{j!} \lim_{t \rightarrow 0} \left( \frac{d}{dt} \right)^j \prod_{k=1}^m (t^2 + \rho_{p_i}(k\tau_{ik})t + 1) \\ &= \sum_{n_1 + \dots + n_m = j} \prod_{k=1}^m \lim_{t \rightarrow 0} \frac{1}{n_k!} \left( \frac{d}{dt} \right)^{n_k} (t^2 + \rho_{p_i}(k\tau_{ik})t + 1) \\ (8) \quad &= \sum_{n_1 + \dots + n_m = j} \prod_{k=1}^m \chi(n_k) \end{aligned}$$

where  $n_1, \dots, n_m$  are integers such that  $0 \leq n_1, \dots, n_m \leq 2$  and

$$\chi(n_k) = \begin{cases} \rho_{p_i}(k\tau_{ik}) & \text{if } n_k = 1 \\ 1 & \text{if } n_k = 0, 2 \end{cases}.$$

Let  $r, j-2r$  be the numbers of 2's and 1's in  $n_1, \dots, n_m$ . If  $2r < j$  and  $n_{s_1} = \dots = n_{s_{j-2r}} = 1$ , we have

$$(8) = \sum_{r=0}^{\lfloor \frac{j}{2} \rfloor} \sum_{1 \leq s_1 < \dots < s_{j-2r} \leq m} \binom{m-j+2r}{r} \rho_{p_i}(k\tau_{is_1}) \cdots \rho_{p_i}(k\tau_{is_{j-2r}})$$

because the number of  $(t_1, \dots, t_r)$  such that  $1 \leq t_1 < \dots < t_r \leq m$  and that  $n_{t_1} = \dots = n_{t_r} = 2$  is  $\binom{m-j+2r}{r}$ . Hence we have

$$(8) = \sum_{r=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{m-j+2r}{r} \sigma_{j-2r}(\rho_{p_i}(k\tau_{i1}), \dots, \rho_{p_i}(k\tau_{im})) + \varepsilon_j \binom{m}{\frac{j}{2}} = C_{ij}^k,$$

because  $\prod_{k=1}^m \chi(n_k) = 1$  if  $2r = j$ ,  $\lfloor \frac{j-1}{2} \rfloor = \lfloor \frac{j}{2} \rfloor - 1$  if  $j$  is even and  $\lfloor \frac{j-1}{2} \rfloor = \lfloor \frac{j}{2} \rfloor$  if  $j$  is odd. Therefore we have

$$\text{Ch}(E_{q_i}^{\ell_1 \dots \ell_m}, g^{r_i k}) = \prod_{j=1}^m (C_{ij}^k)^{\ell_j}. \quad \square$$

EXAMPLE 2.10. When  $m = 1$ , it follows from Proposition 2.9 that

$$\text{Ch}(E_{q_i}^{\ell_1}, g^{r_i}) = (C_{i1}^1)^{\ell_1} = (\sigma_1(\rho_{p_i}(\tau_{i1})))^{\ell_1} = (\rho_{p_i}(\tau_{i1}))^{\ell_1}.$$

Next assume that  $m = 2$  and that  $(\ell_1, \ell_2) \neq (0, 0)$ . Let  $\sigma_k$  the  $k$ -th elementary symmetric polynomial in  $\rho_{p_i}(\tau_{i1}), \rho_{p_i}(\tau_{i2})$ . Then it follows from Proposition 2.9 that

$$\begin{aligned} \text{Ch}(E_{q_i}^{\ell_1, \ell_2}, g^{r_i}) &= (C_{i1}^1)^{\ell_1} (C_{i2}^1)^{\ell_2} = \sigma_1^{\ell_1} (\sigma_2 + 2)^{\ell_2} = \sum_{\kappa_2=0}^{\ell_2} \binom{\ell_2}{\kappa_2} 2^{\ell_2 - \kappa_2} \sigma_1^{\ell_1} \sigma_2^{\kappa_2} \\ &= \sum_{\kappa_2=0}^{\ell_2} \binom{\ell_2}{\kappa_2} 2^{\ell_2 - \kappa_2} (\rho_{p_i}(\tau_{i1}) + \rho_{p_i}(\tau_{i2}))^{\ell_1} (\rho_{p_i}(\tau_{i1}) \rho_{p_i}(\tau_{i2}))^{\kappa_2} \\ &= \sum_{\mu_1=0}^{\ell_1} \sum_{\kappa_2=0}^{\ell_2} \binom{\ell_1}{\mu_1} \binom{\ell_2}{\kappa_2} 2^{\ell_2 - \kappa_2} \rho_{p_i}(\tau_{i1})^{\mu_1 + \kappa_2} \rho_{p_i}(\tau_{i2})^{\ell_1 - \mu_1 + \kappa_2}, \end{aligned}$$

and using the property (7) we can express the value above as a summation of  $\rho_{p_i}$ 's. When  $(\ell_1, \ell_2) = (0, 0)$ , we have

$$\text{Ch}(E_{q_i}^{0,0}, g^{r_i}) = \sigma_1^0 (\sigma_2 + 2)^0 = 1 \implies 2\text{Ch}(E_{q_i}^{0,0}, g^{r_i}) = \rho(0).$$

REMARK 2.11. Let  $h(\ell_1, \dots, \ell_m)$  be a non-negative integer and  $\theta(\ell_1, \dots, \ell_m)$  a natural number defined by

$$h(\ell_1, \dots, \ell_m) = \prod_{j=1}^m \left( \varepsilon_j \binom{m}{\frac{j}{2}} \right)^{\ell_j}, \quad \theta(\ell_1, \dots, \ell_m) = \begin{cases} 2 & (h(\ell_1, \dots, \ell_m) \text{ is odd}) \\ 1 & (h(\ell_1, \dots, \ell_m) \text{ is even}) \end{cases}.$$

It follows from Proposition 2.9 that  $\text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, g^{r_i})$  is expressed as a summation of products of  $\rho_{p_i}(\mu_i)$ 's with integer coefficients and  $h(\ell_1, \dots, \ell_m)$  where  $\mu_i$ 's are non-negative integers. Hence  $\theta(\ell_1, \dots, \ell_m) \text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, g^{r_i})$  is expressed as a summation of  $\rho_{p_i}(\mu)$ 's with integer coefficients because  $2 = \rho(0)$ .

If  $\theta \text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, g^{r_i}) = \sum_s n_s \rho_{p_i}(\mu_{is})$  and hence  $\theta \text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}, (g^{r_i})^{kz}) = \sum_s n_s \rho_{p_i}(kz\mu_{is})$  where  $\theta = 1$  or  $2$ ,  $n_s$ 's are natural numbers,  $z$  is a natural number which is prime to  $p$  and  $\mu_{is}$ 's are non-negative integers, it follows from (3), (4) and (5) that

$$(9) \quad \begin{aligned} \theta I_{D_{\ell_1, \dots, \ell_m}}(g^z) &\equiv \frac{p-1}{p} 2^{m-1} \theta \text{Ch}(E_{q_i}^{\ell_1, \dots, \ell_m}) \widehat{\mathbf{L}}(M)[M] \\ &\quad - \frac{1}{p} \sum_{i=1}^b r_i \sum_s n_s \sum_{\kappa=1}^{p_i-1} \frac{\rho_{p_i}(\kappa z \mu_{is})}{1 - \xi_{p_i}^{-\kappa}} \prod_{j=1}^m \left( -\sqrt{-1} \cot \frac{\pi \kappa z \tau_{ij}}{p_i} \right) \pmod{\mathbb{Z}}. \end{aligned}$$



So we can calculate  $\theta I_{D_{\ell_1, \dots, \ell_m}}(g^z)$  precisely if we can express the value

$$\sum_{k=1}^{p_i-1} \frac{\rho_{p_i}(kz\mu_{is})}{1 - \xi_{p_i}^{-k}} \prod_{j=1}^m \left( -\sqrt{-1} \cot \frac{\pi k z \tau_{ij}}{p_i} \right)$$

as a rational number (see Theorem 3.6).

**Proposition 2.12.** *Let  $p$  be a natural number which is greater than 1 and  $R(x)$  a rational function with real coefficients. Then  $\sum_{k=1}^{p-1} R(\xi_p^k)$  is a real number.*

*Proof.* Since  $R$  is a rational function with real coefficients, we have

$$\overline{\sum_{k=1}^{p-1} R(\xi_p^k)} = \sum_{k=1}^{p-1} \overline{R(\xi_p^k)} = \sum_{k=1}^{p-1} R(\overline{\xi_p^k}) = \sum_{k=1}^{p-1} R(\xi_p^{p-k}) = \sum_{k=1}^{p-1} R(\xi_p^k). \quad \square$$

**Proposition 2.13.** *Let  $p$  be a natural number which is greater than 1,  $a$  an integer and  $b_j$  ( $1 \leq j \leq 2\gamma$ ) natural numbers which are prime to  $p$ . Then we have*

$$\begin{aligned} (-1)^{\gamma+1} \sum_{k=1}^{p-1} \xi_p^{-ka} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma} \frac{\xi_p^{b_j k} + 1}{\xi_p^{b_j k} - 1} &= \sum_{k=1}^{p-1} \sin \frac{2\pi ak}{p} \cot \frac{\pi k}{p} \prod_{j=1}^{2\gamma} \cot \frac{\pi b_j k}{p}, \\ (-1)^\gamma \sum_{k=1}^{p-1} \xi_p^{-ka} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma-1} \frac{\xi_p^{b_j \overline{b_{2\gamma} k}} + 1}{\xi_p^{b_j \overline{b_{2\gamma} k}} - 1} &= \sum_{k=1}^{p-1} \cos \frac{2\pi ak}{p} \cot \frac{\pi k}{p} \prod_{j=1}^{2\gamma-1} \cot \frac{\pi b_j \overline{b_{2\gamma} k}}{p}. \end{aligned}$$

*Proof.* It follows from Proposition 2.12 that the left-hand sides of the equalities above are real numbers. Hence it follows from the equalities below

$$\begin{aligned} (10) \quad \frac{\xi_p^{k\tau} + 1}{\xi_p^{k\tau} - 1} &= \frac{e^{2\pi i k \tau / p} + 1}{e^{2\pi i k \tau / p} - 1} = \frac{i^{-1} \frac{e^{\pi i k \tau / p} + e^{-\pi i k \tau / p}}{2}}{\frac{e^{\pi i k \tau / p} - e^{-\pi i k \tau / p}}{2i}} = -\sqrt{-1} \cot \frac{\pi k \tau}{p} \\ \frac{2}{1 - \xi_p^{-k}} &= \frac{2\xi_p^k}{\xi_p^k - 1} = 1 + \frac{\xi_p^k + 1}{\xi_p^k - 1} = 1 - \sqrt{-1} \cot \frac{\pi k}{p}, \end{aligned}$$

that

$$\begin{aligned} &(-1)^{\gamma+1} \sum_{k=1}^{p-1} \xi_p^{-ka} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma} \frac{\xi_p^{b_j k} + 1}{\xi_p^{b_j k} - 1} \\ &= (-1)^{\gamma+1} \sum_{k=1}^{p-1} \left( \cos \frac{2\pi ak}{p} - \sqrt{-1} \sin \frac{2\pi ak}{p} \right) \left( -\sqrt{-1} \cot \frac{\pi k}{p} \right) \prod_{j=1}^{2\gamma} \left( -\sqrt{-1} \cot \frac{\pi b_j k}{p} \right) \\ &= \sum_{k=1}^{p-1} \sin \frac{2\pi ak}{p} \cot \frac{\pi k}{p} \prod_{j=1}^{2\gamma} \cot \frac{\pi b_j k}{p}, \\ &(-1)^\gamma \sum_{k=1}^{p-1} \xi_p^{-ka} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma-1} \frac{\xi_p^{b_j \overline{b_{2\gamma} k}} + 1}{\xi_p^{b_j \overline{b_{2\gamma} k}} - 1} \\ &= (-1)^\gamma \sum_{k=1}^{p-1} \left( \cos \frac{2\pi ak}{p} - \sqrt{-1} \sin \frac{2\pi ak}{p} \right) \left( -\sqrt{-1} \cot \frac{\pi k}{p} \right) \prod_{j=1}^{2\gamma-1} \left( -\sqrt{-1} \cot \frac{\pi b_j \overline{b_{2\gamma} k}}{p} \right) \end{aligned}$$

$$= \sum_{k=1}^{p-1} \cos \frac{2\pi ak}{p} \cot \frac{\pi k}{p} \prod_{j=1}^{2\gamma-1} \cot \frac{\pi b_j \overline{b_{2\gamma} k}}{p}. \quad \square$$

DEFINITION 2.14. Let  $p$  be a natural number which is greater than 1 and  $b_1, \dots, b_{2n}$  natural numbers which are prime to  $p$ . Then a rational number  $\zeta_j(p; b_1, \dots, b_{2j})$  ( $0 \leq j \leq n$ ) is defined by

$$\begin{aligned} \zeta_j(p; b_1, \dots, b_{2j}) &= (-1)^j 2^{2j} p \sum_{\substack{1 \leq k_1, \dots, k_{2j} < p \\ p | (b_1 k_1 + \dots + b_{2j} k_{2j})}} \left( \left( \frac{k_1}{p} \right) \right) \cdots \left( \left( \frac{k_{2j}}{p} \right) \right) \\ &= \frac{(-1)^j}{p^{2j-1}} \sum_{\substack{1 \leq k_1, \dots, k_{2j} < p \\ p | (b_1 k_1 + \dots + b_{2j} k_{2j})}} (2k_1 - p) \cdots (2k_{2j} - p) \end{aligned}$$

for  $1 \leq j \leq n$  and  $\zeta_0 = -1$ . Set

$$c_k = \langle \overline{b_{2k-1}} b_{2k} \rangle \quad (1 \leq k \leq n), \quad d_k = \langle \overline{b_{2k}} b_{2k+1} \rangle \quad (1 \leq k \leq n-1)$$

and let  $a$  be a non-negative integer. Then an integer  $\eta_j(p; a, b_{2n-2j+1}, \dots, b_{2n})$  ( $0 \leq j \leq n$ ) is defined by  $\eta_0 = 1$ ,  $\eta_j(p; 0, b_{2n-2j+1}, \dots, b_{2n}) = 0$  ( $1 \leq j \leq n$ ) and if  $a > 0$

$$\begin{aligned} \eta_j(p; a, b_{2n-2j+1}, \dots, b_{2n}) &= \sum_{t_{n-1}=0}^{ac_n} \psi(a, c_n; t_{n-1}) \sum_{t_{n-2}=0}^{c_{n-1} d_{n-1} t_{n-1}} \psi(d_{n-1} t_{n-1}, c_{n-1}; t_{n-2}) \\ &\quad \cdots \sum_{t_{n-j}=0}^{c_{n-j+1} d_{n-j+1} t_{n-j+1}} \psi(d_{n-j+1} t_{n-j+1}, c_{n-j+1}; t_{n-j}) \end{aligned}$$

for  $2 \leq j \leq n$  and

$$\eta_1(p; a, b_{2n-1}, b_{2n}) = \sum_{t_{n-1}=0}^{ac_n} \psi(a, c_n; t_{n-1})$$

where  $\psi$  is an integer defined by

$$\psi(a, c; t) = \begin{cases} -1 & (t = ac) \\ 4 \left( \frac{t}{c} - a \right) & (0 < t < ac \text{ and } t \text{ is a multiple of } c) \\ 4 \left( \left[ \frac{t}{c} \right] - a \right) + 2 & (0 < t < ac \text{ and } t \text{ is not a multiple of } c) \\ -2a + 1 & (t = 0) \end{cases}.$$

REMARK 2.15. For  $j > 0$  it follows from Theorem 1.3 that

$$\zeta_j(p; b_1, \dots, b_{2j}) = \sum_{k=1}^{p-1} \cot \frac{\pi b_1 k}{p} \cdots \cot \frac{\pi b_{2j} k}{p}.$$

DEFINITION 2.16. For a non-negative integer  $n$  and natural numbers  $a, c, p, q$ , integers  $\phi_n(a, c)$  and  $\beta(n, p, q)$  are defined by

$$\phi_n(a, c) = \sum_{t=0}^{ac} \psi(a, c; t) t^n, \quad \beta(n, p, q) = \sum_{k=p+1}^q k^n.$$

**Proposition 2.17.** *For natural numbers  $a, c$ , we have*

$$(11) \quad \phi_0(a, c) = -2a^2c,$$

$$(12) \quad \phi_n(a, c) = 4 \sum_{j=1}^{a-1} j \beta(n, jc-1, (j+1)c-1) - 2(2a-1) \beta(n, 0, ac-1) \\ - c^n \{2\beta(n, 0, a-1) + a^n\} \quad (n \geq 1).$$

*Proof.* Since there are  $a-1$   $t$ 's  $\{c, 2c, \dots, (a-1)c\}$  which satisfy the conditions that  $0 < t < ac$  and that  $t$  is a multiple of  $c$ , we have

$$\begin{aligned} \phi_0(a, c) &= \sum_{t=0}^{ac} \psi(a, c; t) = -2a + 1 - 1 + \sum_{t=1}^{ac-1} \left\{ 4 \left( \left\lfloor \frac{t}{c} \right\rfloor - a \right) + 2 \right\} - 2(a-1) \\ &= 2ac(1-2a) + 4 \sum_{t=1}^{ac-1} \left\lfloor \frac{t}{c} \right\rfloor = 2ac(1-2a) + 4 \left( \sum_{j=1}^{a-1} \sum_{t=jc}^{(j+1)c-1} j \right) \\ &= 2ac(1-2a) + 4 \left( \sum_{j=1}^{a-1} cj \right) = 2ac(1-2a) + 4c \frac{a(a-1)}{2} = -2a^2c. \end{aligned}$$

For  $n \geq 1$  we have

$$\begin{aligned} \phi_n(a, c) &= \sum_{t=0}^{ac} \psi(a, c; t) t^n \\ &= (-2a+1) \cdot 0^n + (-1)(ac)^n + \sum_{t=1}^{ac-1} \left\{ 4 \left( \left\lfloor \frac{t}{c} \right\rfloor - a \right) + 2 \right\} t^n - 2 \sum_{j=1}^{a-1} (jc)^n \\ &= -a^n c^n + 4 \sum_{t=1}^{ac-1} t^n \left\lfloor \frac{t}{c} \right\rfloor + (2-4a) \beta(n, 0, ac-1) - 2c^n \beta(n, 0, a-1) \\ &= 4 \left( \sum_{j=1}^{a-1} \sum_{t=jc}^{(j+1)c-1} jt^n \right) - 2(2a-1) \beta(n, 0, ac-1) - c^n \{2\beta(n, 0, a-1) + a^n\} \\ &= 4 \sum_{j=1}^{a-1} j \beta(n, jc-1, (j+1)c-1) - 2(2a-1) \beta(n, 0, ac-1) \\ &\quad - c^n \{2\beta(n, 0, a-1) + a^n\}. \quad \square \end{aligned}$$

**EXAMPLE 2.18.** Using the proposition above, we have

$$\begin{aligned} \eta_1(p; a, b_{2n-1}, b_{2n}) &= \phi_0(a, c_n) = -2a^2c_n, \\ \phi_2(a, c) &= 4 \sum_{j=1}^{a-1} j \beta(2, jc-1, (j+1)c-1) - 2(2a-1) \beta(2, 0, ac-1) - c^2 \{2\beta(2, 0, a-1) + a^2\} \end{aligned}$$

$$\begin{aligned}
&= 4 \sum_{j=1}^{a-1} j \sum_{k=jc}^{(j+1)c-1} k^2 - 2(2a-1) \sum_{k=1}^{ac-1} k^2 - c^2 \left\{ 2 \sum_{k=1}^{a-1} k^2 + a^2 \right\} \\
&= -\frac{1}{3} a^2 c_n (1 + c_n^2) - \frac{1}{3} a^4 c_n^3,
\end{aligned}$$

and hence it follows that

$$\begin{aligned}
\eta_2 &= \eta_2(p; a, b_{2n-3}, \dots, b_{2n}) \\
&= \sum_{t_{n-1}=0}^{ac_n} \psi(a, c_n; t_{n-1}) \sum_{t_{n-2}=0}^{c_{n-1}d_{n-1}t_{n-1}} \psi(d_{n-1}t_{n-1}, c_{n-1}; t_{n-2}) \\
&= \sum_{t_{n-1}=0}^{ac_n} \psi(a, c_n; t_{n-1}) \phi_0(d_{n-1}t_{n-1}, c_{n-1}) \\
&= \sum_{t_{n-1}=0}^{ac_n} \psi(a, c_n; t_{n-1}) (-2c_{n-1}d_{n-1}^2t_{n-1}^2) = (-2c_{n-1}d_{n-1}^2) \phi_2(a, c_n) \\
&= (-2c_{n-1}d_{n-1}^2) \left( -\frac{1}{3} a^2 c_n (1 + c_n^2) - \frac{1}{3} a^4 c_n^3 \right) = \frac{2}{3} a^2 c_{n-1} d_{n-1}^2 c_n (a^2 c_n^2 + c_n^2 + 1).
\end{aligned}$$

### 3. Main results

**Theorem 3.1.** Let  $p$  be a natural number which is greater than 1,  $n$  a natural number,  $\mu$  a non-negative integer and  $b_1, \dots, b_{2n}$  natural numbers which are prime to  $p$ . Then we have

$$\sum_{k=1}^{p-1} \cos \frac{2\pi\mu k}{p} \prod_{j=1}^{2n} \cot \frac{\pi b_j k}{p} \equiv \sum_{k=0}^n \zeta_k(p; b_1, \dots, b_{2k}) \eta_{n-k}(p; a, b_{2k+1}, \dots, b_{2n}) \pmod{p}$$

where  $a = \mu \overline{b_{2n}}$ .

**Proof.** When  $\mu = 0$ , as we see in Remark 2.15, the equality above holds. So we assume that  $\mu > 0$  hereafter. Set

$$a_k = a_k(t_k) = \begin{cases} 0 & (k=0) \\ d_k t_k (= \langle \overline{b_{2k}} b_{2k+1} \rangle t_k) & (1 \leq k \leq n-1) \\ a & (k=n) \end{cases}.$$

For  $\gamma \geq 0$ , set

$$\begin{aligned}
F_\gamma(a, b_1, \dots, b_{2\gamma}) &= \begin{cases} (-1)^{\gamma+1} \sum_{k=1}^{p-1} \xi_p^{-ak} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma} \frac{\xi_p^{b_j k} + 1}{\xi_p^{b_j k} - 1} & (\gamma \geq 1) \\ - \sum_{k=1}^{p-1} \xi_p^{-ak} \frac{\xi_p^k + 1}{\xi_p^k - 1} & (\gamma = 0) \end{cases}, \\
G_\gamma(a, b_1, \dots, b_{2\gamma}) &= \begin{cases} (-1)^\gamma \sum_{k=1}^{p-1} \xi_p^{-ak} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma-1} \frac{\xi_p^{b_j \overline{b_{2\gamma}} k} + 1}{\xi_p^{b_j \overline{b_{2\gamma}} k} - 1} & (\gamma \geq 1) \\ \sum_{k=1}^{p-1} \xi_p^{-ak} & (\gamma = 0) \end{cases},
\end{aligned}$$

$$f_\gamma(a, b_1, \dots, b_{2\gamma}) = \begin{cases} \sum_{k=1}^{p-1} \sin \frac{2\pi ak}{p} \cot \frac{\pi k}{p} \prod_{j=1}^{2\gamma} \cot \frac{\pi b_j k}{p} & (\gamma \geq 1) \\ \sum_{k=1}^{p-1} \sin \frac{2\pi ak}{p} \cot \frac{\pi k}{p} & (\gamma = 0) \end{cases},$$

$$g_\gamma(a, b_1, \dots, b_{2\gamma}) = \begin{cases} \sum_{k=1}^{p-1} \cos \frac{2\pi ak}{p} \cot \frac{\pi k}{p} \prod_{j=1}^{2\gamma-1} \cot \frac{\pi b_j \overline{b_{2\gamma}} k}{p} & (\gamma \geq 1) \\ \sum_{k=1}^{p-1} \cos \frac{2\pi ak}{p} & (\gamma = 0) \end{cases}.$$

Note that the values  $F_\gamma$ ,  $G_\gamma$ ,  $f_\gamma$ ,  $g_\gamma$  are invariant under the substitution  $a \rightarrow a+p$ ,  $b_j \rightarrow b_j+p$ , and that for any integer  $a$  we have

$$G_0(a) = g_0(a) = \begin{cases} p-1 & (\text{if } a \text{ is a multiple of } p) \\ -1 & (\text{if } a \text{ is not a multiple of } p) \end{cases} \equiv -1 \pmod{p}.$$

Note moreover that for a natural number  $\lambda$  which is prime to  $p$ , we have

$$(13) \quad g_\gamma(a, \lambda b_1, \dots, \lambda b_{2\gamma}) = g_\gamma(a, b_1, \dots, b_{2\gamma})$$

because  $\lambda b_j \overline{\lambda b_{2\gamma}} \equiv \lambda \overline{\lambda} b_j \overline{b_{2\gamma}} \equiv b_j \overline{b_{2\gamma}} \pmod{p}$ . As we see in Proposition 2.13,  $F_\gamma$ ,  $G_\gamma$  are real numbers and the following equalities hold for  $\gamma \geq 1$ :

$$F_\gamma(a, b_1, \dots, b_{2\gamma}) = f_\gamma(a, b_1, \dots, b_{2\gamma}),$$

$$G_\gamma(a, b_1, \dots, b_{2\gamma}) = g_\gamma(a, b_1, \dots, b_{2\gamma}).$$

Moreover it follows from Proposition 2.12 and (10) that

$$F_0(a) = - \sum_{k=1}^{p-1} \xi_p^{-ak} \frac{\xi_p^k + 1}{\xi_p^k - 1} = - \sum_{k=1}^{p-1} \left( \cos \frac{2\pi ak}{p} - \sqrt{-1} \sin \frac{2\pi ak}{p} \right) \left( -\sqrt{-1} \cot \frac{\pi k}{p} \right)$$

$$= \sum_{k=1}^{p-1} \sin \frac{2\pi ak}{p} \cot \frac{\pi k}{p} = f_0(a),$$

and similarly that  $G_0(a) = g_0(a)$ .

**Lemma 3.2.** For  $0 \leq k \leq n$  the following equality holds:

$$g_k(0, b_1, \dots, b_{2k}) \equiv \zeta_k(p; b_1, \dots, b_{2k}) \pmod{p}.$$

*Proof.* When  $k = 0$ , we have

$$g_0(0) = \sum_{k=1}^{p-1} \cos 0 = p-1 \equiv \zeta_0 \pmod{p}.$$

For  $1 \leq k \leq n$  since the map  $\overline{b_{2k}}k \rightarrow k$  gives a bijection of  $\mathbb{Z}_p$ , we have

$$g_k(0, b_1, \dots, b_{2k}) = \sum_{k=1}^{p-1} \cot \frac{\pi b_{2k} \overline{b_{2k}} k}{p} \prod_{j=1}^{2k-1} \cot \frac{\pi b_j \overline{b_{2k}} k}{p} = \sum_{k=1}^{p-1} \prod_{j=1}^{2k} \cot \frac{\pi b_j k}{p},$$

which is equal to  $\zeta_k(p; b_1, \dots, b_{2k})$  as we see in Remark 2.15.  $\square$

**Lemma 3.3.** *We have*

$$f_0(a) = \sum_{k=1}^{p-1} \sin \frac{2\pi ak}{p} \cot \frac{\pi k}{p} \equiv -2a \pmod{p}.$$

*Proof.* It follows from the Eisenstein's formula

$$(14) \quad \sum_{k=1}^{p-1} \sin \frac{2\pi ak}{p} \cot \frac{\pi k}{p} = -2p \left( \left( \frac{a}{p} \right) \right).$$

(see [9] p.103 (22)) that

$$f_0(a) = -2p \left( \left( \frac{a}{p} \right) \right) = -2p \left( \frac{a}{p} - \left[ \frac{a}{p} \right] - \frac{1}{2} \right) \equiv -2a \pmod{p}. \quad \square$$

The next Lemma is the key lemma to prove the theorem.

**Lemma 3.4.** *For  $\gamma \geq 1$  we have*

$$g_\gamma(a, b_1, \dots, b_{2\gamma}) = \sum_{t_{\gamma-1}=0}^{ac_\gamma} \psi(a, c_\gamma; t_{\gamma-1}) g_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1, \dots, b_{2\gamma-2}) + g_\gamma(0, b_1, \dots, b_{2\gamma}).$$

*Proof.* Since the map  $b_{2\gamma}k \rightarrow k$  gives a bijection of  $\mathbb{Z}_p$ , for  $\gamma \geq 1$  we have

$$\begin{aligned} & F_\gamma(a+1, b_1, \dots, b_{2\gamma}) - F_\gamma(a, b_1, \dots, b_{2\gamma}) \\ &= (-1)^{\gamma+1} \sum_{k=1}^{p-1} \xi_p^{-k(a+1)} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma} \frac{\xi_p^{b_j k} + 1}{\xi_p^{b_j k} - 1} (1 - \xi_p^k) \\ &= (-1)^\gamma \sum_{k=1}^{p-1} (\xi_p^{-ak} + \xi_p^{-k(a+1)}) \prod_{j=1}^{2\gamma} \frac{\xi_p^{b_j k} + 1}{\xi_p^{b_j k} - 1} \\ &= (-1)^\gamma \sum_{k=1}^{p-1} (\xi_p^{-kab_{2\gamma}} + \xi_p^{-k(a+1)b_{2\gamma}}) \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma-1} \frac{\xi_p^{b_j \overline{b_{2\gamma}k}} + 1}{\xi_p^{b_j \overline{b_{2\gamma}k}} - 1} \\ &= G_\gamma(a \overline{b_{2\gamma}}, b_1, \dots, b_{2\gamma}) + G_\gamma((a+1) \overline{b_{2\gamma}}, b_1, \dots, b_{2\gamma}), \end{aligned}$$

and

$$F_0(a+1) - F_0(a) = (-1)^0 \sum_{k=1}^{p-1} (\xi_p^{-kab_{2\gamma}} + \xi_p^{-k(a+1)b_{2\gamma}}) = G_0(a \overline{b_{2\gamma}}) + G_0((a+1) \overline{b_{2\gamma}}),$$

$$F_\gamma(0, b_1, \dots, b_{2\gamma}) = f_\gamma(0, b_1, \dots, b_{2\gamma}) = 0.$$

Hence it follows that

$$\begin{aligned} & F_\gamma(a, b_1, \dots, b_{2\gamma}) = F_\gamma(a, b_1, \dots, b_{2\gamma}) - F_\gamma(0, b_1, \dots, b_{2\gamma}) \\ &= \sum_{t=0}^{a-1} \{F_\gamma(t+1, b_1, \dots, b_{2\gamma}) - F_\gamma(t, b_1, \dots, b_{2\gamma})\} \\ &= \sum_{t=0}^{a-1} \{G_\gamma(t \overline{b_{2\gamma}}, b_1, \dots, b_{2\gamma}) + G_\gamma((t+1) \overline{b_{2\gamma}}, b_1, \dots, b_{2\gamma})\} \end{aligned}$$

$$(15) \quad = 2 \sum_{i=0}^a G_{\gamma}(t\overline{b_{2\gamma}}, b_1, \dots, b_{2\gamma}) - G_{\gamma}(a\overline{b_{2\gamma}}, b_1, \dots, b_{2\gamma}) - G_{\gamma}(0, b_1, \dots, b_{2\gamma}).$$

Moreover since the map  $b_{2\gamma-1}\overline{b_{2\gamma}k} \rightarrow k$  gives a bijection of  $\mathbb{Z}_p$ , for  $\gamma \geq 1$  we have

$$\begin{aligned} & G_{\gamma}(a+1, b_1, \dots, b_{2\gamma}) - G_{\gamma}(a, b_1, \dots, b_{2\gamma}) \\ &= (-1)^{\gamma} \sum_{k=1}^{p-1} \xi_p^{-k(a+1)} \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma-1} \frac{\xi_p^{b_j \overline{b_{2\gamma}k}} + 1}{\xi_p^{b_j \overline{b_{2\gamma}k}} - 1} (1 - \xi_p^k) \\ &= (-1)^{\gamma+1} \sum_{k=1}^{p-1} (\xi_p^{-ka} + \xi_p^{-k(a+1)}) \frac{\xi_p^{b_{2\gamma-1} \overline{b_{2\gamma}k}} + 1}{\xi_p^{b_{2\gamma-1} \overline{b_{2\gamma}k}} - 1} \prod_{j=1}^{2\gamma-2} \frac{\xi_p^{b_j \overline{b_{2\gamma}k}} + 1}{\xi_p^{b_j \overline{b_{2\gamma}k}} - 1} \\ &= (-1)^{\gamma+1} \sum_{k=1}^{p-1} \left( \xi_p^{-ka\overline{b_{2\gamma-1}b_{2\gamma}}} + \xi_p^{-k(a+1)\overline{b_{2\gamma-1}b_{2\gamma}}} \right) \frac{\xi_p^k + 1}{\xi_p^k - 1} \prod_{j=1}^{2\gamma-2} \frac{\xi_p^{b_j \overline{b_{2\gamma-1}k}} + 1}{\xi_p^{b_j \overline{b_{2\gamma-1}k}} - 1} \\ &= -\left\{ F_{\gamma-1}(a\overline{b_{2\gamma-1}b_{2\gamma}}, b_1\overline{b_{2\gamma-1}}, \dots, b_{2\gamma-2}\overline{b_{2\gamma-1}}) \right. \\ &\quad \left. + F_{\gamma-1}((a+1)\overline{b_{2\gamma-1}b_{2\gamma}}, b_1\overline{b_{2\gamma-1}}, \dots, b_{2\gamma-2}\overline{b_{2\gamma-1}}) \right\} \\ &= -\left\{ F_{\gamma-1}(a\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) + F_{\gamma-1}((a+1)\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \right\}, \end{aligned}$$

where  $b_j^{\gamma} = b_j\overline{b_{2\gamma-1}}$  ( $1 \leq j \leq 2\gamma-2$ ) and hence it follows that

$$\begin{aligned} & G_{\gamma}(a, b_1, \dots, b_{2\gamma}) - G_{\gamma}(0, b_1, \dots, b_{2\gamma}) \\ &= \sum_{s=0}^{a-1} \left\{ G_{\gamma}(s+1, b_1, \dots, b_{2\gamma}) - G_{\gamma}(s, b_1, \dots, b_{2\gamma}) \right\} \\ &= -\sum_{s=0}^{a-1} \left\{ F_{\gamma-1}(s\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) + F_{\gamma-1}((s+1)\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \right\} \\ &= -2 \sum_{s=0}^a F_{\gamma-1}(s\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) + F_{\gamma-1}(a\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \end{aligned}$$

because  $F_{\gamma-1}(0, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) = 0$ . Therefore for  $\gamma \geq 1$  it follows from (15) that

$$\begin{aligned} & G_{\gamma}(a, b_1, \dots, b_{2\gamma}) \\ &= -2 \sum_{s=0}^a F_{\gamma-1}(s\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \\ &\quad + F_{\gamma-1}(a\overline{b_{2\gamma-1}b_{2\gamma}}, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) + G_{\gamma}(0, b_1, \dots, b_{2\gamma}) \\ (16) \quad &= -2 \sum_{s_{\gamma-1}=0}^a F_{\gamma-1}(s_{\gamma-1}c_{\gamma}, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) + F_{\gamma-1}(ac_{\gamma}, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \\ &\quad + G_{\gamma}(0, b_1, \dots, b_{2\gamma}) \\ &\quad \left( \text{Since } t_{\gamma-1}\overline{b_{2\gamma-2}^{\gamma}} \equiv a_{\gamma-1}(t_{\gamma-1}), \quad s_{\gamma-1}c_{\gamma}\overline{b_{2\gamma-2}^{\gamma}} \equiv a_{\gamma-1}(s_{\gamma-1}c_{\gamma}) \pmod{p} \right). \end{aligned}$$

$$\begin{aligned}
&= -2 \sum_{s_{\gamma-1}=0}^a \left\{ 2 \sum_{t_{\gamma-1}=0}^{s_{\gamma-1}c_{\gamma}} G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \right. \\
&\quad \left. - G_{\gamma-1}(a_{\gamma-1}(s_{\gamma-1}c_{\gamma}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) - G_{\gamma-1}(0, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \right\} \\
&\quad + 2 \sum_{t_{\gamma-1}=0}^{ac_{\gamma}} G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \\
&\quad - G_{\gamma-1}(a_{\gamma-1}(ac_{\gamma}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) - G_{\gamma-1}(0, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \\
&\quad + G_{\gamma}(0, b_1, \dots, b_{2\gamma}) \\
(17) \quad &= -4 \sum_{s_{\gamma-1}=0}^a \sum_{t_{\gamma-1}=0}^{s_{\gamma-1}c_{\gamma}} G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \\
&\quad + 2 \sum_{s_{\gamma-1}=0}^a G_{\gamma-1}(a_{\gamma-1}(s_{\gamma-1}c_{\gamma}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) + 2 \sum_{s_{\gamma-1}=0}^a G_{\gamma-1}(0, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \\
&\quad + 2 \sum_{t_{\gamma-1}=0}^{ac_{\gamma}} G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) - G_{\gamma-1}(a_{\gamma-1}(ac_{\gamma}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \\
&\quad - G_{\gamma-1}(0, b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) + G_{\gamma}(0, b_1, \dots, b_{2\gamma}) \\
&= \sum_{s_{\gamma-1}=0}^a \sum_{t_{\gamma-1}=0}^{s_{\gamma-1}c_{\gamma}} \begin{pmatrix} -4 + 2\delta(t_{\gamma-1}, s_{\gamma-1}c_{\gamma}) \\ +2\delta(t_{\gamma-1}, 0) + 2\delta(s_{\gamma-1}, a) \\ -\delta(s_{\gamma-1}, a)\delta(t_{\gamma-1}, s_{\gamma-1}c_{\gamma}) \\ -\delta(s_{\gamma-1}, a)\delta(t_{\gamma-1}, 0) \end{pmatrix} G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) \\
&\quad + g_{\gamma}(0, b_1, \dots, b_{2\gamma}) \\
&= \sum_{t_{\gamma-1}=0}^{ac_{\gamma}} \Lambda(t_{\gamma-1}) G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^{\gamma}, \dots, b_{2\gamma-2}^{\gamma}) + g_{\gamma}(0, b_1, \dots, b_{2\gamma})
\end{aligned}$$

where

$$\Lambda(t) = \sum_{s_{\gamma-1}=N(t)}^a \begin{pmatrix} -4 + 2\delta(t, s_{\gamma-1}c_{\gamma}) \\ +2\delta(t, 0) + 2\delta(s_{\gamma-1}, a) \\ -\delta(s_{\gamma-1}, a)\delta(t, s_{\gamma-1}c_{\gamma}) \\ -\delta(s_{\gamma-1}, a)\delta(t, 0) \end{pmatrix}, \quad N(t) = \begin{cases} \frac{t}{c_{\gamma}} & (\text{if } c_{\gamma}|t) \\ \left\lfloor \frac{t}{c_{\gamma}} \right\rfloor + 1 & (\text{otherwise}) \end{cases}.$$

Here we have

$$\begin{aligned}
\Lambda(ac_{\gamma}) &= -4 + 2 + 0 + 2 - 1 - 0 = -1, \\
\Lambda(t) &= -4 \left( a - \frac{t}{c_{\gamma}} + 1 \right) + 2 + 0 + 2 - 0 - 0 = 4 \left( \frac{t}{c_{\gamma}} - a \right) \\
&\quad \text{if } 0 < t < ac_{\gamma} \text{ and } t \text{ is a multiple of } c_{\gamma}, \\
\Lambda(t) &= -4 \left( a - \left\lfloor \frac{t}{c_{\gamma}} \right\rfloor \right) + 0 + 0 + 2 - 0 - 0 = 4 \left( \left\lfloor \frac{t}{c_{\gamma}} \right\rfloor - a \right) + 2 \\
&\quad \text{if } 0 < t < ac_{\gamma} \text{ and } t \text{ is not a multiple of } c_{\gamma}, \\
\Lambda(0) &= -4(a+1) + 2 + 2(a+1) + 2 - 0 - 1 = -2a + 1,
\end{aligned}$$

which implies that  $\Lambda(t_{\gamma-1})$  is equal to  $\psi(a, c_{\gamma}; t_{\gamma-1})$ . Therefore it follows from (13) that



$$\begin{aligned}
g_\gamma(a, b_1, \dots, b_{2\gamma}) &= G_\gamma(a, b_1, \dots, b_{2\gamma}) \\
&= \sum_{t_{\gamma-1}=0}^{ac_\gamma} \Lambda(t_{\gamma-1}) G_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1^\gamma, \dots, b_{2\gamma-2}^\gamma) + g_\gamma(0, b_1, \dots, b_{2\gamma}) \\
&= \sum_{t_{\gamma-1}=0}^{c_\gamma a} \psi(a, c_\gamma; t_{\gamma-1}) g_{\gamma-1}(a_{\gamma-1}(t_{\gamma-1}), b_1, \dots, b_{2\gamma-2}) + g_\gamma(0, b_1, \dots, b_{2\gamma}). \quad \square
\end{aligned}$$

For  $2 \leq \nu \leq n-1$  set  $\psi_\nu(t_{\nu-1}) = \psi(a_\nu(t_\nu), c_\nu; t_{\nu-1})$ .

It follows from the result of Lemma 3.4 that

$$\begin{aligned}
g_n(a, b_1, \dots, b_{2n}) &= \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) g_{n-1}(a_{n-1}(t_{n-1}), b_1, \dots, b_{2n-2}) + g_n(0, b_1, \dots, b_{2n}) \\
&= \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) \left\{ \sum_{t_{n-2}=0}^{a_{n-1}c_{n-1}} \psi_{n-1}(t_{n-2}) g_{n-2}(a_{n-2}(t_{n-2}), b_1, \dots, b_{2n-4}) \right. \\
&\quad \left. + g_{n-1}(0, b_1, \dots, b_{2n-2}) \right\} \\
&\quad + g_n(0, b_1, \dots, b_{2n}) \\
&= \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) \sum_{t_{n-2}=0}^{a_{n-1}c_{n-1}} \psi_{n-1}(t_{n-2}) g_{n-2}(a_{n-2}(t_{n-2}), b_1, \dots, b_{2n-4}) \\
&\quad + g_{n-1}(0, b_1, \dots, b_{2n-2}) \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) + g_n(0, b_1, \dots, b_{2n}) \\
(18) \quad &= \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) \cdots \sum_{t_2=0}^{a_3c_3} \psi_3(t_2) \sum_{t_1=0}^{a_2c_2} \psi_2(t_1) g_1(a_1(t_1), b_1, b_2) \\
&\quad + \sum_{k=2}^{n-1} g_k(0, b_1, \dots, b_{2k}) \sum_{t_{n-1}=0}^{ac_n} \psi_n(t_{n-1}) \cdots \sum_{t_k=0}^{a_{k+1}c_{k+1}} \psi_{k+1}(t_k) \\
&\quad + g_n(0, b_1, \dots, b_{2n}).
\end{aligned}$$

Here it follows from (16), Example 2.18 and Lemma 3.3 that

$$\begin{aligned}
g_1(a_1, b_1, b_2) &= -2 \sum_{s=0}^{a_1} f_0(sc_1) + f_0(a_1c_1) + g_1(0, b_1, b_2) \\
&\equiv -2 \sum_{s=0}^{a_1} (-2sc_1) - 2a_1c_1 + g_1(0, b_1, b_2) \\
&= 2a_1^2c_1 + g_1(0, b_1, b_2) = -\phi_0(a_1, c_1) + g_1(0, b_1, b_2) \\
&\equiv g_0(0) \sum_{t_0=0}^{a_1c_1} \psi(a_1, c_1; t_0) + g_1(0, b_1, b_2) \pmod{p},
\end{aligned}$$

and hence we have

$$\begin{aligned}
&\sum_{t_{n-1}=0}^{a_n c_n} \psi(a_n, c_n; t_{n-1}) \cdots \sum_{t_1=0}^{a_2 c_2} \psi(a_2, c_2; t_1) g_1(a_1, b_1, b_2) \\
&\equiv g_0(0) \sum_{t_{n-1}=0}^{a_n c_n} \psi(a_n, c_n; t_{n-1}) \cdots \sum_{t_1=0}^{a_2 c_2} \psi(a_2, c_2; t_1) \sum_{t_0=0}^{a_1 c_1} \psi(a_1, c_1; t_0)
\end{aligned}$$

$$\begin{aligned}
& + g_1(0, b_1, b_2) \sum_{t_{n-1}=0}^{a_n c_n} \psi(a_n, c_n; t_{n-1}) \cdots \sum_{t_1=0}^{a_2 c_2} \psi(a_2, c_2; t_1) \\
& \equiv g_0(0) \eta_n(p; a, b_1, \dots, b_{2n}) + g_1(0, b_1, b_2) \eta_{n-1}(p; a, b_3, \dots, b_{2n}) \pmod{p}.
\end{aligned}$$

Moreover we have

$$\begin{aligned}
& \sum_{k=2}^{n-1} g_k(0, b_1, \dots, b_{2k}) \sum_{t_{n-1}=0}^{c_n a} \psi_n(t_{n-1}) \cdots \sum_{t_k=0}^{c_{k+1} a_{k+1}} \psi_{k+1}(t_k) \\
& = \sum_{k=2}^{n-1} g_k(0, b_1, \dots, b_{2k}) \eta_{n-k}(p; a, b_{2k+1}, \dots, b_{2n}),
\end{aligned}$$

and it follows from  $\eta_0(a) = 1$  that

$$g_n(0, b_1, \dots, b_{2n}) = g_n(0, b_1, \dots, b_{2n}) \eta_0(a).$$

Here since  $k \rightarrow \overline{b_{2n}}k$  gives a bijection of  $\mathbb{Z}_p$ , it follows from Lemma 3.2 that

$$\begin{aligned}
& \sum_{k=1}^{p-1} \cos \frac{2\pi\mu k}{p} \prod_{j=1}^{2n} \cot \frac{\pi b_j k}{p} = \sum_{k=1}^{p-1} \cos \frac{2\pi a k}{p} \prod_{j=1}^{2n} \cot \frac{\pi b_j \overline{b_{2n}} k}{p} = g_n(a, b_1, \dots, b_{2n}) \\
& \equiv \sum_{k=0}^n g_k(0, b_1, \dots, b_{2k}) \eta_{n-k}(p; a, b_{2k+1}, \dots, b_{2n}) \\
& \equiv \sum_{k=0}^n \zeta_k(p; b_1, \dots, b_{2k}) \eta_{n-k}(p; a, b_{2k+1}, \dots, b_{2n}) \pmod{p}.
\end{aligned}$$

This completes the proof of Theorem 3.1.  $\square$

**Corollary 3.5.** *Let  $p$  be a natural number which is greater than 1,  $\mu$  a non-negative integer,  $q$  a natural number and  $b_1, \dots, b_{2q}$  natural numbers which are prime to  $p$ . Then*

$$p^{2q-1} \sum_{k=1}^{p-1} \cos \frac{2\pi\mu k}{p} \prod_{j=1}^{2q} \cot \frac{\pi b_j k}{p}$$

*is an integer.*

**Proof.** It follows from the theorem above that

$$\sum_{k=1}^{p-1} \cos \frac{2\pi\mu k}{p} \prod_{j=1}^{2q} \cot \frac{\pi b_j k}{p} = \sum_{k=0}^q \zeta_k(p; b_1, \dots, b_{2k}) \eta_{q-k}(p; \mu \overline{b_{2q}}, b_{2k+1}, \dots, b_{2q}) + p\nu$$

where  $\nu$  is an integer. Here since  $\eta_{q-k}(p; \mu \overline{b_{2q}}, b_{2k+1}, \dots, b_{2q})$  and

$$p^{2q-1} \zeta_k(p; b_1, \dots, b_{2k}) = (-1)^k p^{2q-2k} \sum_{\substack{1 \leq k_1, \dots, k_{2k} < p \\ p \mid (b_1 k_1 + \dots + b_{2k} k_{2k})}} (2k_1 - p) \cdots (2k_{2k} - p)$$

are integers for  $0 \leq k \leq q$ , it follows that

$$p^{2q-1} \sum_{k=1}^{p-1} \cos \frac{2\pi\mu k}{p} \prod_{j=1}^{2q} \cot \frac{\pi b_j k}{p}$$

$$= p^{2q-1} \sum_{j=0}^q \eta_{q-j}(p; \mu \overline{b_{2q}}, b_{2j+1}, \dots, b_{2q}) \zeta_j(p; b_1, \dots, b_{2j}) + p^{2q} \nu$$

is an integer.  $\square$

**Theorem 3.6.** Suppose that the total Chern class of  $TM \otimes \mathbb{C}$  equals  $\prod_{j=1}^m (1 - x_j^2)$  and that  $\theta \text{Ch}(E_{q_i}^{\ell_1 \dots \ell_m}, g^{r_i}) = \sum_s n_s \rho_{p_i}(\mu_{is})$  where  $\theta = \theta(\ell_1 \dots \ell_m) = 1$  or  $2$  (see Remark 2.11),  $n_s$ 's are natural numbers and  $\mu_{is}$ 's are non-negative integers. Let  $a_{is}$ ,  $b_{ij}$  ( $1 \leq i \leq b$ ,  $1 \leq j \leq m$ ),  $h_m$  be integers defined by

$$a_{is} = \begin{cases} \mu_{is} \overline{\tau_{i2n}} & (\text{if } m \text{ is even}) \\ z\mu_{is} & (\text{if } m \text{ is odd}) \end{cases}, \quad b_{ij} = \begin{cases} \overline{z} & (\text{if } m \text{ is odd and } j = 2n) \\ \tau_{ij} & (\text{otherwise}) \end{cases},$$

$$h_m = \begin{cases} 2^{m-1} & (\text{if } m \text{ is even}) \\ 0 & (\text{if } m \text{ is odd}) \end{cases}$$

and  $z$  a natural number which is prime to  $p$ . Then the next equality holds as an element of  $\mathbb{R}/\mathbb{Z}$ .

$$\theta I_{D_{\ell_1, \dots, \ell_m}}(g^z) = \frac{p-1}{p} \theta h_m \prod_{j=1}^m \sigma_j(e^{x_1}, e^{-x_1}, \dots, e^{x_m}, e^{-x_m})^{\ell_j} \prod_{j=1}^m \frac{x_j/2}{\tanh(x_j/2)} [M]$$

$$+ (-1)^{n+1} \frac{1}{p} \sum_{i=1}^b r_i \sum_s n_s \sum_{k=0}^n \zeta_k(p; b_{i1}, \dots, b_{i2k}) \eta_{n-k}(p; a_{is}, b_{i2k+1}, \dots, b_{i2n}).$$

**Proof.** It follows from (10) that

$$(19) \quad \sum_{k=1}^{p-1} \frac{\rho_{p_i}(kz\mu_{is})}{1 - \xi_p^{-k}} \prod_{j=1}^m \left( -\sqrt{-1} \cot \frac{\pi kz\tau_{ij}}{p} \right) = \sum_{k=1}^{p-1} R(\xi_p^k)$$

where

$$R(\xi_p^k) = \frac{\xi_p^{kz\mu_{is}} + \xi_p^{-kz\mu_{is}}}{1 - \xi_p^{-k}} \prod_{j=1}^m \frac{\xi_p^{kz\tau_{ij}} + 1}{\xi_p^{kz\tau_{ij}} - 1}.$$

Then Proposition 2.12 implies that the both sides of (19) are real numbers, and the map  $kz \rightarrow k$  gives a bijection of  $\mathbb{Z}_p$  because  $z$  is prime to  $p$ . Hence we have

$$\begin{aligned} & \sum_{k=1}^{p-1} \frac{\rho_{p_i}(kz\mu_{is})}{1 - \xi_p^{-k}} \prod_{j=1}^m \left( -\sqrt{-1} \cot \frac{\pi kz\tau_{ij}}{p} \right) \\ &= \sum_{k=1}^{p-1} \frac{\xi_p^{kz\mu_{is}} + \xi_p^{-kz\mu_{is}}}{2} \frac{2}{1 - \xi_p^{-k}} \prod_{j=1}^m \left( -\sqrt{-1} \cot \frac{\pi kz\tau_{ij}}{p} \right) \\ &= \sum_{k=1}^{p-1} \cos \frac{2\pi kz\mu_{is}}{p} \left( 1 - \sqrt{-1} \cot \frac{\pi k}{p} \right) \prod_{j=1}^m \left( -\sqrt{-1} \cot \frac{\pi kz\tau_{ij}}{p} \right) \end{aligned}$$

$$\begin{aligned}
&= \begin{cases} (-1)^n \sum_{k=1}^{p-1} \cos \frac{2\pi k z \mu_{is}}{p} \prod_{j=1}^{2n} \cot \frac{\pi k z \tau_{ij}}{p} \\ \quad = (-1)^n \sum_{k=1}^{p-1} \cos \frac{2\pi k \mu_{is}}{p} \prod_{j=1}^{2n} \cot \frac{\pi k \tau_{ij}}{p} & (m = 2n) \\ (-1)^n \sum_{k=1}^{p-1} \cos \frac{2\pi k z \mu_{is}}{p} \cot \frac{\pi k}{p} \prod_{j=1}^{2n-1} \cot \frac{\pi k z \tau_{ij}}{p} \\ \quad = (-1)^n \sum_{k=1}^{p-1} \cos \frac{2\pi k \mu_{is}}{p} \cot \frac{\pi k \bar{z}}{p} \prod_{j=1}^{2n-1} \cot \frac{\pi k \tau_{ij}}{p} & (m = 2n - 1) \end{cases} \\
&= (-1)^n \sum_{k=1}^{p-1} \cos \frac{2\pi \mu_{is} k}{p} \prod_{j=1}^{2n} \cot \frac{\pi b_{jk}}{p}.
\end{aligned}$$

Hence it follows from Theorem 3.1 that

$$\begin{aligned}
(20) \quad & \sum_{k=1}^{p-1} \frac{\rho_{p_i}(kz\mu_{is})}{1 - \xi_p^{-k}} \prod_{j=1}^m \left( -\sqrt{-1} \cot \frac{\pi k z \tau_{ij}}{p} \right) \\
& \equiv (-1)^n \sum_{j=0}^n \zeta_j(p; b_1, \dots, b_{2j}) \eta_{n-j}(p; a_{is}, b_{2j+1}, \dots, b_{2n}).
\end{aligned}$$

Therefore it follows from (3), Proposition 2.6 and (9) that

$$\begin{aligned}
\theta_{I_{D_{\ell_1, \dots, \ell_m}}}(g^z) &= \frac{p-1}{p} \theta h_m \prod_{j=1}^m \sigma_j(e^{x_1}, e^{-x_1}, \dots, e^{x_m}, e^{-x_m})^{\ell_j} \prod_{j=1}^m \frac{x_j/2}{\tanh(x_j/2)} [M] \\
&+ (-1)^{n+1} \frac{1}{p} \sum_{i=1}^b r_i \sum_s n_s \sum_{k=0}^n \zeta_k(p; b_{i1}, \dots, b_{i2k}) \eta_{n-k}(p; a_{is}, b_{i2k+1}, \dots, b_{i2n}).
\end{aligned}$$

This completes the proof of Theorem 3.6.  $\square$

**Corollary 3.7.** Assume that the cyclic group  $\mathbb{Z}_p = \langle g \rangle$  acts on  $2m$ -dimensional closed oriented manifold where  $m$  is even. Then for non-negative integers  $\ell_1, \dots, \ell_m$  we have  $I_{D_{\ell_1, \dots, \ell_m}}(g) = 0$  if  $p$  is odd and  $2\theta I_{D_{\ell_1, \dots, \ell_m}}(g) = 0$  if  $p$  is even.

Proof. Since  $m$  is even,  $\theta I_{D_{\ell_1, \dots, \ell_m}}(g^z)$  does not depend on the natural number  $z$  which is prime to  $p$  because  $a_{is}, b_{ij}$  in the theorem above does not depend on  $z$ . Hence setting  $z = p - 1$  which is prime to  $p$ , we have

$$\begin{aligned}
&\begin{cases} p I_{D_{\ell_1, \dots, \ell_m}}(g) = I_{D_{\ell_1, \dots, \ell_m}}(g^p) = 0, \\ \theta I_{D_{\ell_1, \dots, \ell_m}}(g) = \theta I_{D_{\ell_1, \dots, \ell_m}}(g^{p-1}) = \theta(p-1) I_{D_{\ell_1, \dots, \ell_m}}(g) \end{cases} \\
&\implies \theta(2-p) I_{D_{\ell_1, \dots, \ell_m}}(g) = 2\theta I_{D_{\ell_1, \dots, \ell_m}}(g) = 0,
\end{aligned}$$

which imply that  $I_{D_{\ell_1, \dots, \ell_m}}(g) = 0$  if  $p$  is odd because  $2\theta$  is prime to  $p$ .  $\square$

#### 4. Examples

EXAMPLE 4.1. In this example we consider the case that  $m = 1$ , namely the case that  $M$  is a compact Riemann surface. Let  $p$  be an odd prime number and  $\mathbb{Z}_p$  a cyclic group generated by  $g$ . Then we have  $p_i = p$ , in other words,  $r_i = 1$  for  $1 \leq i \leq b$ . Let  $t_1, \dots, t_b$  be natural numbers such that  $1 \leq t_1, \dots, t_b \leq p-1$ ,  $z$  a natural number which is not a multiple of  $p$ , and set

$$f(t_1, \dots, t_b, z) = \frac{1}{p} \sum_{i=1}^b (2zt_i + \zeta_1(p; t_i, \bar{z})) .$$

Then we have

$$\begin{aligned} f(t_1, \dots, t_b, z) &\equiv \frac{1}{p} \sum_{i=1}^b \left( -2(zt_i)^2 \langle \bar{t}_i \bar{z} \rangle \cdot (-1) + 1 \cdot \zeta_1(p; t_i, \bar{z}) \right) \pmod{\mathbb{Z}} \\ &= \frac{1}{p} \sum_{i=1}^b (\zeta_0 \eta_1(p; zt_i, t_i, \bar{z}) + \zeta_1(p; t_i, \bar{z}) \eta_0) . \end{aligned}$$

If  $\{t_1, \dots, t_b\}$  is the rotation angle of a  $\mathbb{Z}_p$ -action, namely if there exists an action of  $\mathbb{Z}_p = \langle g \rangle$  on a compact Riemann surface such that the rotation angle of  $g$  is  $\{t_1, \dots, t_b\}$ , it follows from Theorem 3.6 that  $f(t_1, \dots, t_b, z) = I_{D_1}(g^z) \in \mathbb{R}/\mathbb{Z}$  because  $\text{Ind}(D_1) = 0$  and  $\text{Ch}(E_{q_i}^1, g) = \rho_p(t_i)$  (see Example 2.10), and hence the following equality holds:

$$(21) \quad f(t_1, \dots, t_b, z) - zf(t_1, \dots, t_b, 1) \equiv 0 \pmod{\mathbb{Z}} .$$

Here as we see in the proof of Proposition 4.6 in [7],  $\{t_1, \dots, t_b\}$  is the rotation angle of a  $\mathbb{Z}_p$ -action if the equality

$$(22) \quad \sum_{i=1}^b \bar{t}_i \equiv 0 \pmod{p}$$

holds. In this example we show that the equality (22) follows from the equality (21), which implies that the equality (21), which corresponds to the equality  $I_{D_1}(g^z) = zI_{D_1}(g)$ , is the necessary and sufficient condition for  $\{t_1, \dots, t_b\}$  to be the rotation angle of a  $\mathbb{Z}_p$ -action.

Here we assume that the equality (21) holds. Since

$$k_1 t_i + k_2 \bar{z} \equiv 0 \iff (k_1 t_i + k_2 \bar{z})z \equiv k_1 t_i z + k_2 \equiv 0 \iff k_2 \equiv -k_1 t_i z \pmod{p} ,$$

it follows that

$$\begin{aligned} \zeta_1(p; t_i, \bar{z}) &= -4p \sum_{\substack{1 \leq k_1, k_2 \leq p \\ p | (t_i k_1 + \bar{z} k_2)}} \left( \left( \frac{k_1}{p} \right) \right) \left( \left( \frac{k_2}{p} \right) \right) = -4p \sum_{k_1=1}^{p-1} \left( \left( \frac{k_1}{p} \right) \right) \left( \left( \frac{-k_1 t_i z}{p} \right) \right) \\ &= 4p \sum_{k=1}^{p-1} \left( \left( \frac{k}{p} \right) \right) \left( \left( \frac{k t_i z}{p} \right) \right) , \end{aligned}$$

and hence we have

$$f(t_1, \dots, t_b, z) = \frac{1}{p} \sum_{i=1}^b \left( 2zt_i + 4p \sum_{k=1}^{p-1} \left( \left( \frac{k}{p} \right) \right) \left( \left( \frac{k t_i z}{p} \right) \right) \right) .$$

Then since

$$\begin{aligned} 2 \left\{ z \left( \left( \frac{kt_i}{p} \right) \right) - \left( \left( \frac{kt_i z}{p} \right) \right) \right\} &= 2z \left( \frac{kt_i}{p} - \left[ \frac{kt_i}{p} \right] - \frac{1}{2} \right) - 2 \left( \frac{kt_i z}{p} - \left[ \frac{kt_i z}{p} \right] - \frac{1}{2} \right) \\ &= 2 \left[ \frac{kt_i z}{p} \right] - 2z \left[ \frac{kt_i}{p} \right] + 1 - z \end{aligned}$$

is an integer and the map  $kt_i \rightarrow k$  gives a bijection of  $\mathbb{Z}_p$ , it follows that

$$\begin{aligned} zf(t_1, \dots, t_b, 1) - f(t_1, \dots, t_b, z) &= 4 \sum_{i=1}^b \sum_{k=1}^{p-1} \left( \left( \frac{k}{p} \right) \right) \left\{ z \left( \left( \frac{kt_i}{p} \right) \right) - \left( \left( \frac{kt_i z}{p} \right) \right) \right\} \\ &= 2 \sum_{i=1}^b \sum_{k=1}^{p-1} \left( \frac{2k}{p} - 1 \right) \left\{ z \left( \left( \frac{kt_i}{p} \right) \right) - \left( \left( \frac{kt_i z}{p} \right) \right) \right\} \\ &= \frac{4}{p} \sum_{i=1}^b \sum_{k=1}^{p-1} k \left\{ z \left( \left( \frac{kt_i}{p} \right) \right) - \left( \left( \frac{kt_i z}{p} \right) \right) \right\} \\ &= \frac{4}{p} \sum_{i=1}^b \sum_{k=1}^{p-1} k \bar{t}_i \left\{ z \left( \left( \frac{k}{p} \right) \right) - \left( \left( \frac{kz}{p} \right) \right) \right\} = \frac{1}{p} T \times F(p, z) \end{aligned}$$

where

$$T = \sum_{i=1}^b \bar{t}_i, \quad F(p, z) = 4 \sum_{k=1}^{p-1} k \left\{ z \left( \left( \frac{k}{p} \right) \right) - \left( \left( \frac{kz}{p} \right) \right) \right\}.$$

Therefore it follows from the assumption that

$$(23) \quad 2p \{2f(t_1, \dots, t_b, 1) - f(t_1, \dots, t_b, 2)\} = T \times 2F(p, 2) \equiv 0 \pmod{p}.$$

Here we have

$$\begin{aligned} 2F(p, 2) &= 8 \sum_{k=1}^{p-1} k \left\{ 2 \left( \left( \frac{k}{p} \right) \right) - \left( \left( \frac{2k}{p} \right) \right) \right\} \\ &= 8 \sum_{k=1}^{p-1} k \left\{ \left( 2 \frac{k}{p} - 0 - 1 \right) - \left( \frac{2k}{p} - \left[ \frac{2k}{p} \right] - \frac{1}{2} \right) \right\} \\ &= 8 \sum_{k=1}^{p-1} k \left[ \frac{2k}{p} \right] - 4 \sum_{k=1}^{p-1} k = 8 \sum_{k=1}^{p-1} k \left[ \frac{2k}{p} \right] - 2p(p-1) \equiv 8 \sum_{k=1}^{p-1} k \left[ \frac{2k}{p} \right] \pmod{p} \\ &= 8 \sum_{k=\frac{p+1}{2}}^{p-1} k = (3p-1)(p-1) \equiv 1 \pmod{p}, \end{aligned}$$

and hence it follows from (23) that  $T \equiv 0 \pmod{p}$ , namely the equality (22) holds.

EXAMPLE 4.2. In this example we consider the case that  $m = 2$ , namely the case that  $M$  is a 4-dimensional closed oriented manifold, which does not necessarily admit an almost complex structure (see Example 3.2 in [7]). In this example using Theorem 3.6, we show that a relationship between the rotation angle of a  $\mathbb{Z}_p$ -action on  $M$  and the signature of  $M$  exists (see (28),(29) below). Let  $g$  be an orientation-preserving periodic diffeomorphism of  $M$  of

order  $p$ . Let  $\sigma(M)$  be the signature of  $M$  and suppose that  $\theta \text{Ch}(E_{q_i}^{\ell_1, \ell_2}, g^{r_i}) = \sum_s n_s \rho_{p_i}(\mu_{is})$  ( $\theta = 1$  or  $2$ ). Then since the first Pontrjagin number  $p_1$  is equal to  $3\sigma(M)$ , it follows from Example 2.7, Example 2.18 and Theorem 3.6 that

$$\begin{aligned}
 \theta I_{D_{\ell_1, \ell_2}}(g) &= \frac{p-1}{2p} \theta 2^{2\ell_1 + \ell_2} 3^{\ell_2} (3\ell_1 + 4\ell_2 + 1) \sigma(M) \\
 &\quad + \sum_{i=1}^b \frac{1}{p_i} \sum_s n_s \sum_{k=0}^1 \zeta_k(p_i; \tau_{i1}, \dots, \tau_{i,2k}) \eta_{1-k}(p; \mu_{is} \overline{\tau_{i2}}, \tau_{i,2k+1}, \tau_{i2}) \\
 &= \frac{p-1}{2p} \theta 2^{2\ell_1 + \ell_2} 3^{\ell_2} (3\ell_1 + 4\ell_2 + 1) \sigma(M) \\
 &\quad + \sum_{i=1}^b \frac{1}{p_i} \sum_s n_s \left\{ (-1) \left( -2 (\mu_{is} \overline{\tau_{i2}})^2 \overline{\tau_{i1}} \tau_{i2} \right) + \zeta_1(p_i; \tau_{i1}, \tau_{i2}) \cdot 1 \right\} \\
 &\iff \\
 (24) \quad p \theta I_{D_{\ell_1, \ell_2}}(g) &= \frac{p-1}{2} \theta 2^{2\ell_1 + \ell_2} 3^{\ell_2} (3\ell_1 + 4\ell_2 + 1) \sigma(M) \\
 &\quad + \sum_{i=1}^b r_i \left( \sum_s 2n_s \mu_{is}^2 \right) \overline{\tau_{i1}} \overline{\tau_{i2}} + \sum_{i=1}^b r_i \left( \sum_s n_s \right) \zeta_1(p_i; \tau_{i1}, \tau_{i2}) \pmod{p}.
 \end{aligned}$$

Here as we see in Example 2.10 we have

$$\text{Ch}(E_{q_i}^{\ell_1, \ell_2}, g) = \sigma_1^{\ell_1} (\sigma_2 + 2)^{\ell_2}$$

where  $\sigma_k$  is the  $k$ -th elementary symmetric polynomial in  $\rho_{p_i}(\tau_{i1}), \rho_{p_i}(\tau_{i2})$ . Hence we have

$$\begin{aligned}
 \text{Ch}(E_{q_i}^{0,0}, g) &= 1 = \frac{1}{2} \rho_{p_i}(0) \iff 2 \text{Ch}(E_{q_i}^{0,0}, g) = \rho_{p_i}(0), \\
 \text{Ch}(E_{q_i}^{1,0}, g) &= \sigma_1 = \rho_{p_i}(\tau_{i1}) + \rho_{p_i}(\tau_{i2}), \\
 \text{Ch}(E_{q_i}^{0,1}, g) &= \sigma_2 + 2 = \rho_{p_i}(\tau_{i1}) \rho_{p_i}(\tau_{i2}) + 2 = \rho_{p_i}(\tau_{i1} + \tau_{i2}) + \rho_{p_i}(\tau_{i1} - \tau_{i2}) + \rho_{p_i}(0).
 \end{aligned}$$

Set  $\lambda = 1$  if  $p$  is odd and  $\lambda = 2$  if  $p$  is even. Then it follows from Corollary 3.7, (24) and the equalities above that

$$\begin{aligned}
 (25) \quad 2\lambda p I_{D_{0,0}}(g) &= \lambda(p-1) \sigma(M) + \lambda \sum_{i=1}^b r_i \zeta_1(p_i; \tau_{i1}, \tau_{i2}) \equiv 0 \pmod{p}, \\
 p \lambda I_{D_{1,0}}(g) &= 8\lambda(p-1) \sigma(M) + 2\lambda \sum_{i=1}^b r_i (\tau_{i1}^2 + \tau_{i2}^2) \overline{\tau_{i1}} \overline{\tau_{i2}} + 2\lambda \sum_{i=1}^b r_i \zeta_1(p_i; \tau_{i1}, \tau_{i2}) \\
 (26) \quad &\equiv -8\lambda \sigma(M) + 2\lambda \sum_{i=1}^b r_i (\tau_{i1} \overline{\tau_{i2}} + \overline{\tau_{i1}} \tau_{i2}) + 2\lambda \sum_{i=1}^b r_i \zeta_1(p_i; \tau_{i1}, \tau_{i2}) \equiv 0 \pmod{p}.
 \end{aligned}$$

Since  $(\tau_{i1} + \tau_{i2})^2 + (\tau_{i1} - \tau_{i2})^2 + 0^2 = 2(\tau_{i1}^2 + \tau_{i2}^2)$ , the next equality also holds.

$$\begin{aligned}
 p \lambda I_{D_{0,1}}(g) \\
 (27) \quad &\equiv -15\lambda \sigma(M) + 4\lambda \sum_{i=1}^b r_i (\tau_{i1} \overline{\tau_{i2}} + \overline{\tau_{i1}} \tau_{i2}) + 3\lambda \sum_{i=1}^b r_i \zeta_1(p_i; \tau_{i1}, \tau_{i2}) \equiv 0 \pmod{p}.
 \end{aligned}$$

Here we have

$$\begin{cases} 2 \times (25) - (26) \equiv 6\lambda\sigma(M) - 2\lambda \sum_{i=1}^b r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) \equiv 0 \pmod{p} \\ 3 \times (25) - (27) \equiv 12\lambda\sigma(M) - 4\lambda \sum_{i=1}^b r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) \equiv 0 \pmod{p} \end{cases}$$

$$\iff 2\lambda \sum_{i=1}^b r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) \equiv 6\lambda\sigma(M) \pmod{p},$$

and since 2 is prime to  $p$  if  $p$  is odd, it follows that

$$(28) \quad 4 \sum_{i=1}^b r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) \equiv 12\sigma(M) \pmod{p} \quad \text{if } p \text{ is even,}$$

$$(29) \quad \sum_{i=1}^b r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) \equiv 3\sigma(M) \pmod{p} \quad \text{if } p \text{ is odd.}$$

For example, we consider the case that  $M = \mathbb{CP}^2$ . Let  $p$  be a prime number which is greater than 3 and  $a, b$  natural numbers such that  $1 < a < b < p$ . Then

$$\mathbb{CP}^2 \ni q = [z_0 : z_1 : z_2] \rightarrow g \cdot q = [z_0 : \xi_p^a z_1 : \xi_p^b z_2] \in \mathbb{CP}^2$$

defines an action of the cyclic group  $\mathbb{Z}_p = \langle g \rangle$ . The fixed point set of  $g^k$  ( $1 \leq k \leq p-1$ ) consists of  $q_1 = [1 : 0 : 0]$ ,  $q_2 = [0 : 1 : 0]$ ,  $q_3 = [0 : 0 : 1]$ , and the rotation angle of  $g$  is

$$((\tau_{11}, \tau_{12}), (\tau_{21}, \tau_{22}), (\tau_{31}, \tau_{32})) = ((a, b), (p-a, b-a), (p-b, p+a-b)).$$

Then  $r_i = 1$  for  $1 \leq i \leq 3$  and we have

$$\begin{aligned} \sum_{i=1}^3 r_i(\tau_{i1}\overline{\tau_{i2}} + \overline{\tau_{i1}}\tau_{i2}) &\equiv a\overline{b} + \overline{a}b - a\overline{(b-a)} + \overline{-a}(b-a) - b\overline{(a-b)} - \overline{b}(a-b) \\ &\equiv a\overline{b} + \overline{a}b - a\overline{(b-a)} - \overline{a}b + 1 + b\overline{(b-a)} - \overline{b}a + 1 \\ &= (b-a)\overline{(b-a)} + 1 + 1 \equiv 3 = 3\sigma(\mathbb{CP}^2) \pmod{p}. \end{aligned}$$

On the other hand since  $3 \equiv 3\sigma(M) \iff 3(\sigma(M) - 1) \equiv 0 \pmod{p}$ ,  $M$  does not admit an  $\mathbb{Z}_p$ -action with the rotation angle above if  $\sigma(M) - 1$  is not a multiple of  $p$ .

For example let  $p$  be any prime number such that  $p+3$  is a multiple of 4 and set  $n = (p+3)/2$ . Let  $M$  be the connected sum of  $n$  copies of  $\mathbb{CP}^2$  and  $n-2$  copies of  $\mathbb{CP}^2$  which is the underlying manifold of  $\mathbb{CP}^2$  with the orientation reversed. Then the Euler number  $e(M)$  of  $M$  is equal to  $2n$  and the equality (2) is satisfied for  $b = 3$ . But since  $\sigma(M) - 1 = n - (n-2) - 1 = 1$  is not a multiple of  $p$ ,  $M$  does not admit an  $\mathbb{Z}_p$ -action with the rotation angle above. Note that  $M$  does not admit any almost complex structure because  $e(M) + \sigma(M) = 2n + 2$  is not a multiple of 4 (see [8] p.1625).

EXAMPLE 4.3. Let  $M$  be a 6-dimensional closed oriented manifold, which does not necessarily admit an almost complex structure (see Example 3.3 in [7]). In this example using Theorem 3.6, we show that  $M$  does not admit the  $\mathbb{Z}_6$ -action of isotropy orders

$$(p_1, p_2, p_3, \dots, p_{v+2}) = (6, 3, 2, \dots, 2).$$



Here we assume that  $M$  admits a  $\mathbb{Z}_6$ -action of isotropy orders above. Then it follows from the equality (2) that the Euler number  $e(M)$  of  $M$  satisfies the equality:

$$\sum_{i=1}^{\nu+2} r_i - e(M) = 1 + 2 + 3\nu - e(M) = 3 \left( \nu + 1 - \frac{e(M)}{3} \right) \equiv 0 \pmod{6},$$

which implies that there does not exist such a  $\mathbb{Z}_6$ -action above unless the following condition is satisfied:

$$(30) \quad e(M) \text{ is a multiple of 3 and } \nu + 1 - \frac{e(M)}{3} \text{ is an even number.}$$

So we assume that the condition above is satisfied.

Let  $g$  be a generator of  $\mathbb{Z}_6$  and  $\{\tau_{i1}, \tau_{i2}, \tau_{i3}\}$  the rotation angle of  $g^i$ . Note that  $\tau_{1j} = 1$  or 5 and  $\tau_{2j} = 1$  or 2 for  $1 \leq j \leq 3$  and hence the mod.  $p_i$ -inverse of  $\tau_{ij}$  is equal to  $\tau_{ij}$  itself for  $i = 1, 2$ . Here it follows from Proposition 2.9 that

$$\text{Ch}(E_{q_i}^{0,0,1}, g) = C_{i3}^1 = \rho_{p_i}(\tau_{i1} \pm \tau_{i2} \pm \tau_{i3}) + 2 \sum_{1 \leq j \leq 3} \rho_{p_i}(\tau_{ij}) = \sum_s n_s \rho_{p_i}(\mu_{is})$$

where

$$\begin{aligned} (n_1, \mu_{i1}) &= (1, \tau_{i1} + \tau_{i2} + \tau_{i3}), & (n_2, \mu_{i2}) &= (1, \tau_{i1} + \tau_{i2} - \tau_{i3}), \\ (n_3, \mu_{i3}) &= (1, \tau_{i1} - \tau_{i2} + \tau_{i3}), & (n_4, \mu_{i4}) &= (1, \tau_{i1} - \tau_{i2} - \tau_{i3}), \\ (n_5, \mu_{i5}) &= (2, \tau_{i1}), & (n_6, \mu_{i6}) &= (2, \tau_{i2}), & (n_7, \mu_{i7}) &= (2, \tau_{i3}). \end{aligned}$$

Then since  $\text{Ind}(D_{0,0,1})$  vanishes as we see in Remark 2.2 and  $\zeta_0 = -1$ ,  $\eta_0 = 1$ , it follows from Theorem 3.6 that

$$(31) \quad I_{D_{0,0,1}}(g^z) = -\frac{1}{p} \sum_{i=1}^{\nu+2} r_i \sum_s n_s F_s(i, z)$$

where

$$F_s(i, z) = -\eta_2(p_i; z\mu_{is}, \tau_{i1}, \tau_{i2}, \tau_{i3}, \bar{z}) + \zeta_1(p_i; \tau_{i1}, \tau_{i2})\eta_1(p_i; z\mu_{is}, \tau_{i3}, \bar{z}) + \zeta_2(p_i; \tau_{i1}, \tau_{i2}, \tau_{i3}, \bar{z}).$$

Here it follows from Example 2.18 for  $n = 2$  that

$$\begin{aligned} \zeta_1 &= \zeta_1(p_i; \tau_{i1}, \tau_{i2}) = -\frac{1}{p_i} \sum_{\substack{1 \leq k_1, k_2 < p_i \\ p_i | (\tau_{i1}k_1 + \tau_{i2}k_2)}} (2k_1 - p_i)(2k_2 - p_i), \\ \zeta_2 &= \zeta_2(p_i; \tau_{i1}, \tau_{i2}, \tau_{i3}, z) = \frac{1}{p_i^3} \sum_{\substack{1 \leq k_1, k_2, k_3, k_4 < p_i \\ p_i | (\tau_{i1}k_1 + \tau_{i2}k_2 + \tau_{i3}k_3 + \bar{z}k_4)}} (2k_1 - p_i)(2k_2 - p_i)(2k_3 - p_i)(2k_4 - p_i), \\ \eta_1 &= \eta_1(p_i; z\mu_{is}, \tau_{i3}, \bar{z}) = -2a_{is}^2 c_2 = -2(z\mu_{is})^2 \langle \overline{\tau_{i3}} \bar{z} \rangle, \\ \eta_2 &= \eta_2(p_i; z\mu_{is}, \tau_{i1}, \tau_{i2}, \tau_{i3}, \bar{z}) = \frac{2}{3} a_{is}^2 c_1 d_1^2 c_2 (a_{is}^2 c_2^2 + c_2^2 + 1) \\ &= \frac{2}{3} (z\mu_{is})^2 \langle \overline{\tau_{i1}} \tau_{i2} \rangle \langle \overline{\tau_{i2}} \tau_{i3} \rangle \langle \overline{\tau_{i3}} \bar{z} \rangle \left\{ (z\mu_{is})^2 \langle \overline{\tau_{i3}} \bar{z} \rangle^2 + \langle \overline{\tau_{i3}} \bar{z} \rangle^2 + 1 \right\}. \end{aligned}$$

When  $i = 1$ ,  $p_i = 6$ ,  $z = 1$ ,  $(\tau_{11}, \tau_{12}, \tau_{13}) = (1, 1, 1)$ , we have

$$\mu_{11} = 1 + 1 + 1 = 3, \quad \mu_{12} = 1 + 1 - 1 = 1, \quad \mu_{13} = 1 - 1 + 1 = 1, \quad \mu_{14} = 1 - 1 - 1 = -1,$$

$$\mu_{15} = \mu_{16} = \mu_{17} = 1, \quad \bar{z} = 1,$$

and direct computation shows that

$$\begin{aligned}\zeta_1(6; \tau_{11}, \tau_{12}) &= \zeta_1(6; 1, 1) = \frac{20}{3}, \\ \zeta_2(6; \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \zeta_2(6; 1, 1, 1, 1) = \frac{164}{9}, \\ \eta_1(6; z\mu_{11}, \tau_{13}, \bar{z}) &= \eta_1(6; 3, 1, 1) = -18, \\ \eta_1(6; z\mu_{14}, \tau_{13}, \bar{z}) &= \eta_1(6; -1, 1, 1) = -2, \\ \eta_1(6; z\mu_{1s}, \tau_{13}, \bar{z}) &= \eta_1(6; 1, 1, 1) = -2 \quad (s = 2, 3, 5, 6, 7), \\ \eta_2(6; z\mu_{11}, \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \eta_2(6; 3, 1, 1, 1, 1) = 66, \\ \eta_2(6; z\mu_{14}, \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \eta_2(6; -1, 1, 1, 1, 1) = 2, \\ \eta_2(6; z\mu_{1s}, \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \eta_2(6; 1, 1, 1, 1, 1) = 2 \quad (s = 2, 3, 5, 6, 7),\end{aligned}$$

and hence we have

$$\sum_s n_s F_s(1, 1) = \sum_{s=1}^4 \left\{ (-1)u_{1s} + \frac{20}{3}v_{1s} + \frac{164}{9} \right\} + 2 \sum_{s=5}^7 \left\{ (-1)u_s + \frac{20}{3}v_s + \frac{164}{9} \right\} = -\frac{1276}{9}$$

where  $(u_{11}, v_{11}) = (66, -18)$ ,  $(u_{1s}, v_{1s}) = (2, -2)$  ( $s = 2, 3, 4, 5, 6, 7$ ).

When  $i = 2$ ,  $p_i = 3$ ,  $z = 1$ ,  $(\tau_{21}, \tau_{22}, \tau_{23}) = (1, 1, 2)$ , we have

$$\begin{aligned}\mu_{21} &= 1 + 1 + 2 = 4, \quad \mu_{22} = 1 + 1 - 2 = 0, \quad \mu_{23} = 1 - 1 + 2 = 2, \quad \mu_{24} = 1 - 1 - 2 = -2, \\ \mu_{25} &= \mu_{26} = 1, \quad \mu_{27} = 2,\end{aligned}$$

and direct computation shows that

$$\begin{aligned}\zeta_1(3; \tau_{21}, \tau_{22}) &= \zeta_1(3; 1, 1) = \frac{2}{3}, \\ \zeta_2(3; \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \zeta_2(3; 1, 1, 2, 1) = -\frac{2}{9}, \\ \eta_1(3; z\mu_{21}, \tau_{23}, \bar{z}) &= \eta_1(3; 4, 2, 1) = -64, \\ \eta_1(3; z\mu_{22}, \tau_{23}, \bar{z}) &= \eta_1(3; 0, 2, 1) = 0, \\ \eta_1(3; z\mu_{2s}, \tau_{13}, \bar{z}) &= \eta_1(3; 2, 2, 1) = -16 \quad (s = 3, 7), \\ \eta_1(3; z\mu_{24}, \tau_{13}, \bar{z}) &= \eta_1(3; -2, 2, 1) = -16, \\ \eta_1(3; z\mu_{2s}, \tau_{13}, \bar{z}) &= \eta_1(3; 1, 2, 1) = -4 \quad (s = 5, 6), \\ \eta_2(3; z\mu_{21}, \tau_{21}, \tau_{22}, \tau_{23}, \bar{z}) &= \eta_2(3; 4, 1, 1, 2, 1) = 5888, \\ \eta_2(3; z\mu_{22}, \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \eta_2(3; 0, 1, 1, 2, 1) = 0, \\ \eta_2(3; z\mu_{2s}, \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \eta_2(3; 2, 1, 1, 2, 1) = 448 \quad (s = 3, 7), \\ \eta_2(3; z\mu_{24}, \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \eta_2(3; -2, 1, 1, 2, 1) = 448, \\ \eta_2(3; z\mu_{2s}, \tau_{11}, \tau_{12}, \tau_{13}, \bar{z}) &= \eta_2(3; 1, 1, 1, 2, 1) = 48 \quad (s = 5, 6),\end{aligned}$$

and hence we have

$$\sum_s n_s F_s(2, 1) = \sum_{s=1}^4 \left\{ (-1)u_{2s} + \frac{2}{3}v_{2s} - \frac{2}{9} \right\} + 2 \sum_{s=5}^7 \left\{ (-1)u_{2s} + \frac{2}{3}v_{2s} - \frac{2}{9} \right\} = -\frac{71732}{9}$$

where  $(u_{21}, v_{21}) = (5888, -64)$ ,  $(u_{22}, v_{22}) = (0, 0)$ ,  $(u_{2s}, v_{2s}) = (448, -16)$  ( $s = 3, 4, 7$ ),  $(u_{2s}, v_{2s}) = (48, -4)$  ( $s = 5, 6$ ).

When  $3 \leq i \leq \nu + 2$ ,  $p_i = 2$  and it follows from Remark 2.3 and (20) that

$$\sum_s n_s F_s(i, 1) = 0.$$

Hence it follows from (5) and (31) that

$$\begin{aligned} I_{D_{0,0,1}}(g) &= -\frac{1}{6} \sum_{i=1}^{\nu+2} r_i \sum_s n_s F_s(i, z) \\ &= \frac{1}{6} \left( \frac{1276}{9} + 2 \cdot \frac{71732}{9} + 0 \right) = \frac{72370}{27} = \frac{10}{27} \in \mathbb{R}/\mathbb{Z}. \end{aligned}$$

Using the same method, we have

$$I_{D_{0,0,1}}(g^5) = \frac{17}{27} \in \mathbb{R}/\mathbb{Z},$$

and therefore it follows that

$$I_{D_{0,0,1}}(g^5) - 5I_{D_{0,0,1}}(g) = \frac{7}{9} \in \mathbb{R}/\mathbb{Z}.$$

Namely the equality

$$I_{D_{0,0,1}}(g^5) = 5I_{D_{0,0,1}}(g)$$

does not hold for  $(\tau_{11}, \tau_{12}, \tau_{13}; \tau_{21}, \tau_{22}, \tau_{23}) = (1, 1, 1; 1, 1, 2)$ .

The direct computation using the same argument shows that the equality

$$I_{D_{0,0,1}}(g^5) = 5I_{D_{0,0,1}}(g)$$

does not hold for  $\tau_{1j} = 1, 5$  ( $1 \leq j \leq 3$ ) and  $\tau_{2j} = 1, 2$  ( $1 \leq j \leq 3$ ). This is a contradiction and therefore  $M$  does not admit the  $\mathbb{Z}_6$ -action of isotropy orders above.

For example let  $N$  be a 4-dimensional closed oriented manifold,  $R$  a compact Riemann surface of genus  $r$  such that  $r \equiv 1 \pmod{3}$  and set  $M = N \times R$ . Then the Euler number of  $M$  is a multiple of 6 because the Euler number of  $R$  is  $2(1 - r)$ , and the condition (30) is satisfied for any odd natural number  $\nu$ . Then it follows from the result above that  $M$  does not admit the  $\mathbb{Z}_6$ -action of isotropy orders  $(p_1, p_2, p_3, \dots, p_{\nu+2}) = (6, 3, 2, \dots, 2)$ .

REMARK 4.4. In the example above,  $M$  is an almost complex manifold if  $N$  is an almost complex manifold. But the  $\mathbb{Z}_6$ -action does not necessarily preserve the almost complex structure and the result above can not be obtained from the methods in [7], where the action is assumed to preserve the almost complex structure.

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